

ON CONFORMAL QC GEOMETRY, SPHERICAL QC MANIFOLDS AND CONVEX COCOMPACT SUBGROUPS OF $\mathrm{Sp}(n+1, 1)$

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ABSTRACT. Conformal qc geometry of spherical qc manifolds are investigated. We construct the qc Yamabe operators on qc manifolds, which are covariant under the conformal qc transformations. A qc manifold is scalar positive, negative or vanishing if and only if its qc Yamabe invariant is positive, negative or zero, respectively. On a scalar positive spherical qc manifold, we can construct the Green function of the qc Yamabe operator, which can be applied to construct a conformally invariant tensor. It becomes a spherical qc metric if the qc positive mass conjecture is true. Conformal qc geometry of spherical qc manifolds can be applied to study convex cocompact subgroups of $\mathrm{Sp}(n+1, 1)$. On a spherical qc manifold constructed from such a discrete subgroup, we construct a spherical qc metric of Nayatani type. As a corollary, we prove that such a spherical qc manifold is scalar positive, negative or vanishing if and only if the Poincaré critical exponent of the discrete subgroup is less than, greater than or equal to $2n+2$, respectively.

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1. INTRODUCTION

A locally conformally flat manifold is a Riemannian manifold which is locally conformally equivalent to the Euclidean space. Its complex counterpart is the spherical CR manifold, which is a pseudohermitian manifold locally conformally equivalent to the Heisenberg group. A conformal class of the locally conformally flat manifolds (or spherical CR manifolds) can be described as a manifold whose coordinate charts are given by open subsets of the Euclidean space (or the Heisenberg group) and elements of $SO(n+1, 1)$ (or $SU(n+1, 1)$) as transition maps. In this paper, we will investigate their quaternionic counterpart: the spherical qc manifolds.

The notion of a qc manifold was introduced by Biquard [3] as the quaternionic counterpart of the notion of a pseudohermitian manifold in the CR geometry. A qc manifold is denoted by (M, g, \mathbb{Q}) , where M is a $(4n+3)$ -dimensional manifold, g is the Carnot-Carathéodory metric on a codimension three distribution H and \mathbb{Q} is a rank-three bundle of quaternionic structures. As the Tanaka-Webster connection in the CR geometry, the Biquard connection is the canonical connection for qc manifolds [3]. (M, g, \mathbb{Q}) is said to be conformal to $(M, \tilde{g}, \mathbb{Q})$, if $\tilde{g} = \phi g$ for some positive function ϕ . A qc manifold (M, g, \mathbb{Q}) is called *spherical* if it is locally conformally qc equivalent to an open set of the quaternionic Heisenberg group. We are interested in the conformal classes of spherical qc manifolds. A conformal class can be described as a manifold whose coordinate charts are given by open subsets of the quaternionic Heisenberg group and elements of $Sp(n+1, 1)$ as transition maps. This is a topological description of a conformal class of spherical qc manifolds. The purpose of this paper is to investigate conformal qc geometry for spherical qc manifolds.

Spherical qc manifolds are abundant. We can construct spherical qc manifolds by taking connected sums. The connected sum of two spherical qc manifolds is constructed as follows (cf. § 2.5). Suppose that $M_{(1)}$ and $M_{(2)}$ are two spherical qc manifolds with one puncture $\eta_i \in M_{(i)}$, $i = 1, 2$, each. Let U_1 and U_2 be neighborhoods of η_1 and η_2 , respectively. Now identify U_i with the ball of quaternionic Heisenberg group with radius 2, centered at origin. We remove the closed balls with radius $t < 1$. Let $U_i(t, 1)$, $i = 1, 2$, be corresponding rings with inner radius t and outer radius 1. There exist conformal qc transformations mapping $U_1(t, 1)$ to $U_2(t, 1)$, which identify the inner boundary of $U_1(t, 1)$ with the outer boundary $U_2(t, 1)$ and vice versa. We glue M_1 and M_2 by such a transformation to get a new spherical qc manifold.

As in the locally conformally flat case and the spherical CR case, convex cocompact subgroups of $Sp(n+1, 1)$ provide lots of examples of spherical qc manifolds. $Sp(n+1, 1)$ acts isometrically on the unit ball B^{4n+4} of the $(n+1)$ -dimensional quaternionic space, when the ball is equipped with the quaternionic hyperbolic metric. Also $Sp(n+1, 1)$ acts conformally on its boundary, the sphere S^{4n+3} with the standard qc metric. Let Γ be a convex cocompact subgroup of $Sp(n+1, 1)$. The *limit set* of Γ is

$$\Lambda(\Gamma) = \overline{\Gamma q} \cap S^{4n+3},$$

for $q \in B^{4n+4}$, where $\overline{\Gamma q}$ is the closure of the orbit of q under Γ , and

$$\Omega(\Gamma) = S^{4n+3} \setminus \Lambda(\Gamma)$$

is the maximal open set where Γ acts discontinuously. It is known that $\Omega(\Gamma)/\Gamma$ is a compact spherical qc manifold when Γ is a convex cocompact subgroup of $\mathrm{Sp}(n+1, 1)$. Modelled on the theory of Riemann surfaces, the basic problems for locally conformally flat manifolds are their classification and the moduli space. It is so for spherical CR manifolds and spherical qc manifolds. It is well known that each compact Riemann surface has a metric with constant curvature, which only depends on the conformal class of Riemann surfaces. This invariant metric was generalized by Habermann-Jost [11] [12] to the locally conformally flat case. They constructed a canonical metric on each scalar positive locally conformally flat manifold, which only depends on the conformal class of such manifolds. This construction was generalized to the spherical CR case by the second author [33].

The uniformization problem is to show each scalar positive spherical qc manifold can be constructed from some convex cocompact subgroup of $\mathrm{Sp}(n+1, 1)$. The problem was solved in [32] for locally conformally flat manifolds and in [4] for spherical CR manifolds. See [20] and [35] for discussion of their moduli spaces.

In Section 2, we recall the definitions of a qc manifold, the Biquard connection and the ball model of the quaternionic hyperbolic space. For our purpose later, we describe explicitly the flat qc structure on the quaternionic Heisenberg group and the standard spherical qc metric on the sphere. The group $\mathrm{Sp}(n+1, 1)$ is the group of all $(n+2) \times (n+2)$ quaternionic matrices which preserve the following hyperhermitian form:

$$(1.1) \quad Q(q, p) = -q_1 \bar{p}_1 - \cdots - q_{n+1} \bar{p}_{n+1} + q_{n+2} \bar{p}_{n+2},$$

where $q = (q_1, \dots, q_{n+2})$, $p = (p_1, \dots, p_{n+2}) \in \mathbb{H}^{n+2}$. In this paper we only consider left quaternionic vector spaces. So a matrix acts on a vector from right. $\mathrm{Sp}(n+1, 1)$ acts isometrically on the ball model of the quaternionic hyperbolic space and conformally on the sphere with the standard qc structure. We write down such a conformal action explicitly by calculating the conformal factor in terms of matrix elements of an element of $\mathrm{Sp}(n+1, 1)$. We also show that a qc manifold (M, g, \mathbb{Q}) is spherical if and only if it is a manifold with coordinates charts $\{(U_i, \phi_i)\}$, where $\phi_i : U_i \rightarrow S^{4n+3}$ and transition maps are given by the induced action of elements of $\mathrm{Sp}(n+1, 1)$.

Since topologically defined spherical qc manifolds depends on the conformal class of qc metrics, we need to study conformal qc geometry. In particular, we are interested in constructing conformal invariants. We will construct conformally covariant operators, since some quantity associated to such a operator will provide us a conformal invariant. In this way we can obtain invariants for topologically defined spherical qc manifolds. Given a qc manifold (M, g, \mathbb{Q}) , let $\Delta_{g, \mathbb{Q}}$ be the SubLaplacian operator and let $s_{g, \mathbb{Q}}$ be its scalar curvature. In Section 3, we construct the *qc Yamabe operator*

$$L_{g, \mathbb{Q}} = b_n \Delta_{g, \mathbb{Q}} + s_{g, \mathbb{Q}},$$

where $b_n = 4 \frac{Q+2}{Q-2}$, and $Q = 4n+6$ is the homogeneous dimension of M . This is a conformally covariant operator. Namely, under the conformal change

$$(1.2) \quad \tilde{g} = \phi^{\frac{4}{Q-2}} g,$$

it satisfies the following transformation law

$$L_{\tilde{g}, \mathbb{Q}} f = \phi^{-\frac{Q+2}{Q-2}} L_{g, \mathbb{Q}}(\phi f),$$

for any smooth real function f . Denote by $G_{g,\mathbb{Q}}(\xi, \cdot)$ the Green function of the qc Yamabe operator with the pole at ξ , i.e.

$$L_{g,\mathbb{Q}}G_{g,\mathbb{Q}}(\xi, \cdot) = \delta_\xi,$$

where δ_ξ is the Dirac function at the point ξ .

As in the Riemannian case [32] and the CR case [33], we will see that for a connected compact qc manifold (M, g, \mathbb{Q}) , one and only one of the following cases holds: there is a qc metric \tilde{g} conformal to g which have either positive, negative or zero scalar curvature everywhere. The manifold is called scalar positive, scalar negative or scalar vanishing respectively. This is equivalent to its first eigenvalue of the qc Yamabe operator or the qc Yamabe invariant is positive, negative or zero, respectively. This is a property of conformal classes. On a scalar positive spherical qc manifold, the Green function of the qc Yamabe operator $L_{g,\mathbb{Q}}$ always exists, and

$$(1.3) \quad \rho_{g,\mathbb{Q}}(\xi, \eta) = \frac{1}{\phi(\xi)\phi(\eta)} \cdot \frac{C_Q}{\|\xi^{-1}\eta\|^{\frac{4}{Q-2}}}, \quad \xi, \eta \in \mathcal{H}^n,$$

is its singular part, if we identify a neighborhood of ξ with an open set of the quaternionic Heisenberg group with the qc metric $g = \phi^{\frac{4}{Q-2}}g_0$. Here g_0 is the standard qc metric on the quaternionic Heisenberg group \mathcal{H}^n , $\|\cdot\|$ is the norm on the quaternionic Heisenberg group and C_Q is a positive constant (3.12). Moreover the limit

$$(1.4) \quad \mathcal{A}_{g,\mathbb{Q}}(\xi) = \lim_{\eta \rightarrow \xi} |G_{g,\mathbb{Q}}(\xi, \eta) - \rho_{g,\mathbb{Q}}(\xi, \eta)|^{\frac{1}{Q-2}}$$

of the nonsingular part of the Green function exists, and $\mathcal{A}_{g,\mathbb{Q}}^2 g$ is conformal invariant, i.e.

$$\mathcal{A}_{\tilde{g},\mathbb{Q}}^2 \tilde{g} = \mathcal{A}_{g,\mathbb{Q}}^2 g,$$

under the conformal transformation (1.2). We also prove the transformation law of Green functions under the conformal change (1.2):

$$G_{\tilde{g},\mathbb{Q}}(\xi, \eta) = \frac{1}{\phi(\xi)\phi(\eta)} G_{g,\mathbb{Q}}(\xi, \eta).$$

As in the conformal geometry and the CR geometry, we propose the following positive mass conjecture.

The qc positive mass conjecture: *Let (M, g, \mathbb{Q}) be a compact scalar positive spherical qc manifold with $\dim M = 4n + 3$. Then,*

1. *For each $\xi \in M$, there exists a local qc diffeomorphism C_ξ from a neighborhood of ξ to the quaternionic Heisenberg group \mathcal{H}^n such that $C_\xi(\xi) = \infty$ and*

$$\left(C_\xi^{-1}\right)^* \left(G_{g,\mathbb{Q}}(\xi, \cdot)^{\frac{4}{Q-2}} g\right) = h^{\frac{4}{Q-2}} g_0,$$

where

$$h(\eta) = 1 + A_{g,\mathbb{Q}}(\xi) \|\eta\|^{-Q+2} + O(\|\eta\|^{-Q+1}),$$

near ∞ , and g_0 is the standard qc metric on \mathcal{H}^n . $A_{g,\mathbb{Q}}(\xi)$ is called the qc mass at the point ξ .

2. $A_{g,\mathbb{Q}}(\xi)$ is nonnegative and is zero if and only if (M, g, \mathbb{Q}) is qc equivalent to the standard sphere.

Li gave the statement of the CR positive mass theorem in [25]. A complete proof of CR positive mass theorem was given by Cheng, Chiu and Yang in [4] recently.

This conjecture implies that $\mathcal{A}_{g,\mathbb{Q}}$ is non-vanishing, so $\mathcal{A}_{g,\mathbb{Q}}^2 g$ is a conformally invariant qc metric. By estimating the qc Yamabe invariant of the connected sum, we prove that some

connected sum of two scalar positive spherical qc manifolds is also scalar positive. So scalar positive spherical qc manifolds are abundant.

In Section 5, we recall the definitions of the convex cocompact subgroups of $\mathrm{Sp}(n+1, 1)$ and the Patterson-Sullivan measure. The *Poincaré critical exponent* $\delta(\Gamma)$ of a discrete subgroup Γ is defined as

$$\delta(\Gamma) = \inf \left\{ s > 0; \sum_{\gamma \in \Gamma} e^{-\frac{1}{2}s \cdot d(p, \gamma(q))} < \infty \right\},$$

where p and q are two points in the ball B^{4n+4} and $d(\cdot, \cdot)$ is the quaternionic hyperbolic distance on B^{4n+4} . $\delta(\Gamma)$ is independent of the particular choice of points p and q . Fix a point $q \in B^{4n+4}$, the series

$$\sum_{\gamma \in \Gamma} e^{-\frac{s}{2} \cdot d(p, \gamma(q))}$$

converges for $s > \delta(\Gamma)$ and any $p \in B^{4n+4}$, and diverges for any $s < \delta(\Gamma)$. For any convex cocompact subgroup Γ of $\mathrm{Sp}(n+1, 1)$, there exists a probability measure μ_Γ supported on its limit set $\Lambda(\Gamma)$, called the *Patterson-Sullivan measure*, such that

$$\gamma^* \mu_\Gamma = |\gamma'|^{\delta(\Gamma)} \mu_\Gamma$$

for any $\gamma \in \Gamma$ (cf. [6]), where $|\gamma'|$ is the conformal factor.

The conformal qc geometry of spherical qc manifolds can be applied to study convex cocompact subgroups of $\mathrm{Sp}(n+1, 1)$. Let $G_S(\xi, \cdot)$ be the Green function of the qc Yamabe operator with the pole at ξ on the standard sphere S^{4n+3} . In Section 6, by integrating Green function with respect to the Patterson-Sullivan measure, we define a C^∞ function on the open set $\Omega(\Gamma)$:

$$\phi_\Gamma(\xi) = \left(\int_{\Lambda(\Gamma)} G_S^\kappa(\xi, \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{1}{\kappa}}, \quad \kappa = \frac{2\delta(\Gamma)}{Q-2}.$$

Then

$$g_\Gamma = \phi_\Gamma^{\frac{4}{Q-2}} g$$

is invariant under Γ , and so defines a metric on $\Omega(\Gamma)/\Gamma$. This is the qc generalization of Nayatani's canonical metric in conformal geometry [26]. We prove that if $\delta(\Gamma) < 2n+2$ (resp. $\delta(\Gamma) > 2n+2$, resp. $\delta(\Gamma) = 2n+2$), then the scalar curvature of $(\Omega(\Gamma)/\Gamma, g_\Gamma, \mathbb{Q})$ is positive (resp. negative, resp. zero) everywhere. This result was proved for locally conformally flat manifolds by Nayatani [26]. For spherical CR manifolds it was proved by Nayatani [27] and Wang [33] independently.

In the Appendix, we give a simple proof of the Green function of the qc Yamabe operator on the quaternionic Heisenberg group.

2. QC MANIFOLDS

2.1. Qc manifolds. A *quaternionic contact (qc) manifold* (M, g, \mathbb{Q}) is a $(4n+3)$ -dimensional manifold M with a codimension three distribution H locally given as the kernel of a \mathbb{R}^3 -valued 1-form $\Theta = (\theta_1, \theta_2, \theta_3)$, on which g is a Carnot-Carathéodory metric. In addition, H has an $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure, that is, it is equipped with a rank-three bundle

$$\mathbb{Q} = \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\},$$

which consists of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the commuting relation of quaternions:

$$(2.1) \quad I_1 I_2 = -I_2 I_1 = I_3, \quad I_1^2 = I_2^2 = I_3^2 = -id|_H.$$

They are hermitian compatible with the metric:

$$(2.2) \quad g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot),$$

and satisfy the compatibility condition

$$(2.3) \quad g(I_s X, Y) = d\theta_s(X, Y),$$

for any $X, Y \in H, s = 1, 2, 3$. We denote $\mathbb{I} := (I_1, I_2, I_3)$.

We say (M, g, \mathbb{Q}) is *conformal* to $(M, \tilde{g}, \mathbb{Q})$ if $\tilde{g} = \phi^{\frac{4}{Q-2}} g$ for some smooth positive function ϕ on M . The conformal class of qc manifolds is denoted by $(M, [g], \mathbb{Q})$. In the definition of qc manifolds, the \mathbb{R}^3 valued 1-form Θ is unique up to a rotation by the following lemma.

Lemma 2.1. (cf. p. 100 in [15]) *Let (M, g, \mathbb{Q}) be a qc manifold. If Θ and Θ' are two compatible \mathbb{R}^3 -valued 1-form such that $\text{Ker}(\Theta) = \text{Ker}(\Theta') = H$ locally, then we have $\Theta' = \Psi\Theta$ for some $\Psi \in \text{SO}(3)$ -valued smooth functions Ψ .*

Proof. Write $\Theta = (\theta_1, \theta_2, \theta_3)$ and $\Theta' = (\theta'_1, \theta'_2, \theta'_3)$. The condition $\text{Ker}(\Theta) = \text{Ker}(\Theta') = H$ implies that $\theta'_s = \sum_{t=1}^3 \Psi_{st} \theta_t$ for some $\text{GL}(3)$ -valued function (Ψ_{st}) . Applying the exterior derivative, we find that $d\theta'_s = \sum_{t=1}^3 (d\Psi_{st} \wedge \theta_t + \Psi_{st} d\theta_t)$, which restricted to H gives $g(I'_s X, Y) = \sum_{t=1}^3 \Psi_{st} g(I_t X, Y)$ for any $X, Y \in H$. Consequently we have

$$(2.4) \quad I'_s = \sum_{t=1}^3 \Psi_{st} I_t.$$

So we must have $(\Psi_{st}) \in \text{SO}(3)$. □

Given a qc manifold (M, g, \mathbb{Q}) , there exists a canonical connection defined by Biquard in [3] when $\dim M > 7$, and by Duchemin in [7] for the 7-dimensional case. It is called *Biquard connection* now.

Theorem 2.1. (cf. §2.1.A. in [3]) *If $(M^{4n+3}, g, \mathbb{Q})$ (for $\dim M > 7$) has a qc structure and g is a Carnot-carathéodory metric on H , then there exists a unique connection ∇ on H and a unique supplementary subspace V of H in TM , such that*

- (i) ∇ preserves the decomposition $H \oplus V$ and the metric;
- (ii) for $X, Y \in H$, one has $T_{X,Y} = -[X, Y]_V$;
- (iii) ∇ preserves the $\text{Sp}(n)\text{Sp}(1)$ -structure on H ;
- (iv) for $R \in V$, the endomorphism $\cdot \rightarrow (T_{R,\cdot})_H$ of H lies in the orthogonal of $\mathfrak{sp}_n \oplus \mathfrak{sp}_1$;
- (v) the connection on V is induced by the natural identification of V with the subspace \mathfrak{sp}_1 of the endomorphisms of H .

A mapping $F : (\tilde{M}, \tilde{g}, \tilde{\mathbb{Q}}) \rightarrow (M, g, \mathbb{Q})$ is called a *conformal qc mapping* if

$$(2.5) \quad F^* g = \phi \tilde{g}, \quad F^* \mathbb{Q} = \tilde{\mathbb{Q}},$$

for some positive function ϕ . The pull back $F^* A$ of an endomorphism A of H is defined as

$$(2.6) \quad F^* A(X) := F_*^{-1} [A(F_* X)]$$

for any $X \in H$. If F is invertible and F and F^{-1} are both qc mappings, F is called a *qc diffeomorphism*. If f is a qc diffeomorphism, we must have $f_*\tilde{H} = H$, where H and \tilde{H} are horizontal subbundles of TM and $T\tilde{M}$, respectively.

2.2. The quaternionic Heisenberg group. The simplest qc manifold is the *quaternionic Heisenberg group* $\mathcal{H}^n = \mathbb{H}^n \oplus \text{Im}\mathbb{H}$, whose multiplication is given by

$$(y, t) \cdot (y', t') = (y + y', t + t' + 2\text{Im}(yy')),$$

where $y, y' \in \mathbb{H}^n$ and $t = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$, $t' = t'_1\mathbf{i} + t'_2\mathbf{j} + t'_3\mathbf{k} \in \text{Im}\mathbb{H}$. The conjugation of a quaternion number $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ is $x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}$. The neutral element is $(0, 0)$ and the inverse of (y, t) is $(-y, -t)$. The norm of the quaternionic Heisenberg group \mathcal{H}^n is defined by

$$(2.7) \quad \|(y, t)\| := (|y|^4 + |t|^2)^{\frac{1}{4}}.$$

We have the following automorphisms of \mathcal{H}^n :

(1) *dilations*:

$$(2.8) \quad D_\delta : (y, t) \longrightarrow (\delta y, \delta^2 t), \quad \delta > 0;$$

(2) *left translations*:

$$(2.9) \quad \tau_{(y', t')} : (y, t) \longrightarrow (y', t') \cdot (y, t);$$

(3) *rotations*:

$$(2.10) \quad U : (y, t) \longrightarrow (yU, t), \quad \text{for } U \in \text{Sp}(n),$$

where

$$(2.11) \quad \text{Sp}(n) = \{U \in \text{GL}(n, \mathbb{H}) | U\bar{U}^t = I_n\};$$

(4) The *inversion*:

$$(2.12) \quad R : (y, t) \longrightarrow \left(-(|y|^2 - t)^{-1}y, \frac{-t}{|y|^4 + |t|^2} \right);$$

(5) $\text{Sp}(1)$ acts on \mathcal{H}^n as:

$$(2.13) \quad \sigma : (y, t) \longrightarrow (\sigma y, \sigma t \sigma^{-1}),$$

where the action on the first factor is left multiplication by $\sigma \in \mathbb{H}$ with $|\sigma| = 1$, while the action on the second factor is isomorphism with $\text{SO}(3)$.

Note that for $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ and $x' = x'_1 + x'_2\mathbf{i} + x'_3\mathbf{j} + x'_4\mathbf{k}$, we have

$$(2.14) \quad \begin{aligned} \text{Im}(xx') &= \text{Im}\{(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})(x'_1 - x'_2\mathbf{i} - x'_3\mathbf{j} - x'_4\mathbf{k})\} \\ &= (-x_1x'_2 + x_2x'_1 - x_3x'_4 + x_4x'_3)\mathbf{i} + (-x_1x'_3 + x_3x'_1 + x_2x'_4 - x_4x'_2)\mathbf{j} \\ &\quad + (-x_1x'_4 + x_4x'_1 - x_2x'_3 + x_3x'_2)\mathbf{k} =: \sum_{k,j=1}^4 b_{kj}^s x_k x'_j \mathbf{i}_s \end{aligned}$$

(cf. (2.15) in [36]), where b_{kj}^s is the (k, j) -th entry of the following matrices b^s :

$$(2.15) \quad \begin{aligned} b^1 &:= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & b^2 &:= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ b^3 &:= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to see that matrices b^1, b^2, b^3 satisfy the commuting relation of quaternions:

$$(2.16) \quad (b^1)^2 = (b^2)^2 = (b^3)^2 = -id, \quad b^1 b^2 b^3 = -id.$$

By (2.14), the multiplication of the quaternionic Heisenberg group in terms of real variables can written as (cf. [36])

$$(y, t) \cdot (y', t') = \left(y + y', t_s + t'_s + 2 \sum_{l=0}^{n-1} \sum_{j,k=1}^4 b_{kj}^s y_{4l+k} y'_{4l+j} \right),$$

where $s = 1, 2, 3$, $y = (y_1, y_2, \dots, y_{4n})$, $y' = (y'_1, y'_2, \dots, y'_{4n}) \in \mathbb{R}^{4n}$, $t = (t_1, t_2, t_3)$, $t' = (t'_1, t'_2, t'_3) \in \mathbb{R}^3$. We denote

$$(2.17) \quad Y_{4l+j} := \frac{\partial}{\partial y_{4l+j}} + 2 \sum_{s=1}^3 \sum_{k=1}^4 b_{kj}^s y_{4l+k} \frac{\partial}{\partial t_s},$$

$l = 0, \dots, n-1$, $j = 1, \dots, 4$. They are left invariant vector fields on the quaternionic Heisenberg group \mathcal{H}^n .

The horizontal subspace $H_0 := \text{span}\{Y_1, \dots, Y_{4n}\}$ generates the corresponding Lie algebra of the quaternionic Heisenberg group. The standard \mathbb{R}^3 -valued contact form of the group is

$$(2.18) \quad 2\Theta_0 := dt - y \cdot d\bar{y} + dy \cdot \bar{y}.$$

If we write $\Theta_0 = (\theta_{0;1}, \theta_{0;2}, \theta_{0;3})$, then we have

$$(2.19) \quad 2\theta_{0;s} = dt_s - 2 \sum_{l=0}^{n-1} \sum_{j,k=1}^4 b_{kj}^s y_{4l+k} dy_{4l+j}, \quad s = 1, 2, 3,$$

by using (2.14) again. Then $\text{Ker}\Theta_0 = H_0$. The standard Carnot-Carathéodory metric on the group is defined as

$$g_0(Y_\alpha, Y_\beta) = 2\delta_{\alpha\beta},$$

for $\alpha, \beta = 1, \dots, 4n$. We set $\mathbb{Q}_0 := \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\}$, where transformations I_s , $s = 1, 2, 3$, on H_0 are given by

$$I_s Y_{4l+k} = \sum_{j=1}^4 b_{jk}^s Y_{4l+j},$$

for $l = 0, \dots, n-1$, $k = 1, 2, 3, 4$. It is direct to check I_1, I_2 and I_3 satisfying the commuting relation of quaternions in (2.1). Recall that the wedge product of 1-forms ω_1 and ω_2 is given by

$$(2.20) \quad (\omega_1 \wedge \omega_2)(X, Y) := \omega_1(X)\omega_2(Y) - \omega_1(Y)\omega_2(X),$$

for any vector field X and Y . It is easy to see that

$$\begin{aligned} d\theta_{0;s}(Y_{4l'+k}, Y_{4l+j}) &= - \sum_{a=0}^{n-1} \sum_{j', k'=1}^4 b_{k'j'}^s dy_{4a+k'} \wedge dy_{4a+j'}(Y_{4l'+k}, Y_{4l+j}) \\ &= -2b_{kj}^s \delta_{ll'} = g_0(I_s Y_{4l'+k}, Y_{4l+j}), \end{aligned}$$

since b^s is antisymmetric. Thus, g_0 is compatible with Θ_0 . So $(\mathcal{H}^n, g_0, \mathbb{Q}_0)$ is a qc manifold.

2.3. The quaternionic hyperbolic space and the standard qc structure on the sphere.

The *quaternionic projective space* $\mathbb{H}P^{n+1}$ of dimension $n+1$ is the set of left quaternionic lines in \mathbb{H}^{n+2} . More precisely,

$$\mathbb{H}P^{n+1} := (\mathbb{H}^{n+2} \setminus \{0\}) / \sim,$$

where \sim is the equivalent relation: $(p'_1, \dots, p'_{n+2}) \sim (q'_1, \dots, q'_{n+2}) \in \mathbb{H}^{n+2}$ if there is a non-zero quaternion number λ such that

$$(p'_1, \dots, p'_{n+2}) = (\lambda q'_1, \dots, \lambda q'_{n+2}).$$

Let $P : \mathbb{H}^{n+2} \setminus \{0\} \rightarrow \mathbb{H}P^{n+1}$ be the canonical projection onto the quaternionic projective space. Under the induced action of $\mathrm{Sp}(n+1, 1)$ on $\mathbb{H}P^{n+1}$, there are three invariant subsets

$$\begin{aligned} D_+ &:= \{q' \in \mathbb{H}P^{n+1}; Q(q', q') > 0\}, \\ D_0 &:= \{q' \in \mathbb{H}P^{n+1}; Q(q', q') = 0\}, \\ D_- &:= \{q' \in \mathbb{H}P^{n+1}; Q(q', q') < 0\}. \end{aligned}$$

Then, as a homogeneous space for $\mathrm{Sp}(n+1, 1)$, D_+ is equivalent to the *quaternionic hyperbolic space*. In this case we must have $q_{n+2} \neq 0$. So a point in D_+ is equivalent to $(q, 1)$ for some $q \in \mathbb{H}^{n+1}$, i.e. $(q'_{n+2} q'_1, \dots, q'_{n+2} q'_{n+1}, 1)$.

We introduce a positive definite hyperhermitian form on \mathbb{H}^{n+1} :

$$\langle q, p \rangle := q_1 \bar{p}_1 + \dots + q_{n+1} \bar{p}_{n+1}.$$

It is obvious that $U \in \mathrm{Sp}(n+1)$ (cf. (2.11)) if and only if $\langle qU, pU \rangle = \langle q, p \rangle$ for any $q, p \in \mathbb{H}^{n+1}$. We have the ball model for quaternionic hyperbolic space:

$$B^{4n+4} = \{q \in \mathbb{H}^{n+1}; \langle q, q \rangle < 1\}.$$

Let $\gamma = (\gamma_{ij}) \in \mathrm{Sp}(n+1, 1)$. γ is a $(n+2) \times (n+2)$ matrix acts on left quaternionic vector space \mathbb{H}^{n+2} from right. For $q \in \mathbb{H}^{n+1}$, $(q, 1)$ is a vector in \mathbb{H}^{n+2} . The right action of γ on this vector is denoted by $(q, 1)\gamma$, whose l -th component is

$$[(q, 1)\gamma]_l := \sum_{m=1}^{n+1} q_m \gamma_{ml} + \gamma_{(n+2)l}, \quad l = 1, \dots, n+2.$$

Note that

$$([(q, 1)\gamma]_1, \dots, [(q, 1)\gamma]_{n+2}) \sim ([[(q, 1)\gamma]_{n+2}^{-1} [(q, 1)\gamma]_1, \dots, [(q, 1)\gamma]_{n+2}^{-1} [(q, 1)\gamma]_{n+1}, 1),$$

where $[(q, 1)\gamma]_{n+2} \neq 0$ by

$$|[(q, 1)\gamma]_1|^2 + \cdots + |[(q, 1)\gamma]_{n+1}|^2 - |[(q, 1)\gamma]_{n+2}|^2 = 0,$$

for $q \in S^{4n+3}$. So $\mathrm{Sp}(n+1, 1)$ induces an action on S^{4n+3} by

$$(2.21) \quad \gamma(q) := ([(q, 1)\gamma]_{n+2}^{-1} [(q, 1)\gamma]_1, \dots, [(q, 1)\gamma]_{n+2}^{-1} [(q, 1)\gamma]_{n+1}) \quad \text{for } q \in S^{4n+3}.$$

$\mathrm{Sp}(n+1, 1)$ also induces an action on B^{4n+4} in this way.

The *fundamental invariant* on the unit ball is given by

$$(2.22) \quad (q, p) = \frac{1 - \langle q, p \rangle}{(1 - |q|^2)^{\frac{1}{2}} (1 - |p|^2)^{\frac{1}{2}}},$$

and

$$(2.23) \quad |(q, p)| = \cosh \left(\frac{1}{2} d(q, p) \right),$$

for $q, p \in B^{4n+4}$, where $d(q, p)$ is the *quaternionic hyperbolic distance* between q and p , which is invariant under the action of $\mathrm{Sp}(n+1, 1)$ (cf. p. 523 in [6]).

For a \mathbb{H} -valued function $\mathbf{f} = f_1 + f_2\mathbf{i} + f_3\mathbf{j} + f_4\mathbf{k}$, we set

$$d\mathbf{f} := \sum_{l=1}^{4n+4} \frac{\partial \mathbf{f}}{\partial x_l} dx_l = df_1 + df_2\mathbf{i} + df_3\mathbf{j} + df_4\mathbf{k}.$$

It is easy to see that for two \mathbb{H} -valued function \mathbf{f} and \mathbf{g} , we have

$$d(\mathbf{f} \cdot \mathbf{g}) = d\mathbf{f} \cdot \mathbf{g} + \mathbf{f} \cdot d\mathbf{g}.$$

In particular, we have $dq = dx_0 + dx_1\mathbf{i} + dx_2\mathbf{j} + dx_3\mathbf{k}$ if we write $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. For a \mathbb{H}^n -valued function $\mathbf{f} = (f_1, \dots, f_n)$, we write $d\mathbf{f} := (df_1, \dots, df_n)$.

For a point $\zeta = (\zeta_1, \dots, \zeta_{n+1}) \in S^{4n+3} = \{\xi \in \mathbb{H}^{n+1} : |\xi| = 1\}$, we consider a quaternionic subspace of the tangent space:

$$H_\zeta := \{v \in \mathbb{H}^{n+1} : \langle v, \zeta \rangle = 0\}.$$

It is easy to see that H_ζ is a left quaternionic subspace of \mathbb{H} -dimension n . Then $H = \cup_{\zeta \in S^{4n+3}} H_\zeta$ is the standard horizontal bundle of the tangent bundle of the sphere. Let $\eta : [0, 1] \rightarrow S^{4n+3}$ be any smooth curve such that $\eta(0) = \zeta$. We identify the vector $(\eta'_1(0), \dots, \eta'_{n+1}(0))$ with a tangential vector $X_0 = \sum_{j=1}^{4n+4} v_j \frac{\partial}{\partial x_j}$ at point ζ , if we write $\eta'_l(0) = v_{4l-3} + v_{4l-2}\mathbf{i} + v_{4l-1}\mathbf{j} + v_{4l}\mathbf{k}$. The standard $\mathrm{Im}\mathbb{H}$ -valued contact form on S^{4n+3} is given by

$$(2.24) \quad \Theta_S = \sum_{l=1}^{n+1} (d\zeta_l \cdot \bar{\zeta}_l - \zeta_l \cdot d\bar{\zeta}_l),$$

where $\zeta = (\zeta_1, \dots, \zeta_{n+1}) \in S^{4n+3}$. If we write $\zeta_l = x_{4l-3} + x_{4l-2}\mathbf{i} + x_{4l-1}\mathbf{j} + x_{4l}\mathbf{k}$, $l = 1, \dots, n+1$, and $\Theta_S = (\theta_1, \theta_2, \theta_3)$, then we have

$$\theta_s = -2 \sum_{l=0}^n \sum_{k,j=1}^4 b_{kj}^s x_{4l+k} dx_{4l+j},$$

by using identity (2.14) again, and so

$$d\theta_s = -2 \sum_{l=0}^n \sum_{k,j=1}^4 b_{kj}^s dx_{4l+k} \wedge dx_{4l+j}.$$

The transformation I_s on H_ζ is given by left multiplying \mathbf{i}_s :

$$(\eta'_1(0), \dots, \eta'_{n+1}(0)) \mapsto (\mathbf{i}_s \eta'_1(0), \dots, \mathbf{i}_s \eta'_{n+1}(0)).$$

We can check that

$$(2.25) \quad \mathbf{i}_s(x_1 + x_2 \mathbf{i} + x_3 \mathbf{j} + x_4 \mathbf{k}) = \sum_{j,k=1}^4 b_{jk}^s x_k \mathbf{i}_{j-1}$$

(here $\mathbf{i}_0 = 1$) (cf. (2.2) in [37]). We define

$$I_s \partial_{x_{4l+k}} := \sum_{j=1}^4 b_{jk}^s \partial_{x_{4l+j}},$$

for $l = 0, \dots, n-1$, $j = 1, 2, 3, 4$, $s = 1, 2, 3$. Then for $v \in H_\zeta$, we have $\langle I_s v, \zeta \rangle = 0$, i.e. $I_s v \in H_\zeta$. $\mathbb{Q}_s = \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\}$ is a $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -structure on H . Denote by g_S the restriction to the horizontal subspace H of the Euclidean metric on S^{4n+3} multiplying a factor 4. We can prove that g_S is compatible to Θ_S and \mathbb{Q}_S on S^{4n+3} as in the case of the quaternionic Heisenberg group, i.e. for any $X, Y \in H$,

$$g_S(I_s X, Y) = d\theta_s(X, Y), \quad s = 1, 2, 3.$$

The group of conformal qc transformations of S^{4n+3} consists of quaternionic fractional linear transformation $\mathrm{Sp}(n+1, 1)/\text{center}$ [17]. We have the following qc Liouville type theorem.

Theorem 2.2. (Qc Liouville type theorem) (cf. Theorem 8.5 in [17]) *Every conformal qc transformation between open subsets of S^{4n+3} is the restriction of a global conformal qc transformation.*

We can identify \mathcal{H}^n with the boundary Σ of the *Siegel domain* in \mathbb{H}^{n+1} ,

$$(2.26) \quad \Sigma := \{(q, q_{n+1}) \in \mathbb{H}^n \times \mathbb{H} : \mathrm{Re} q_{n+1} = |q|^2\},$$

by using the projection

$$\begin{aligned} \hat{\pi} : \quad \Sigma &\longrightarrow \mathcal{H}^n, \\ (q, q_{n+1}) &\longmapsto (q, |q|^2 - q_{n+1}). \end{aligned}$$

The *Cayley transform* is the map from the sphere S^{4n+3} minus the southern point to the quadratic hypersurface Σ defined by

$$\hat{F} : S^{4n+3} \longrightarrow \Sigma, \quad (\zeta, \zeta_{n+1}) \longmapsto (q, q_{n+1}),$$

where

$$(2.27) \quad q = (1 + \zeta_{n+1})^{-1} \zeta, \quad q_{n+1} = (1 + \zeta_{n+1})^{-1} (1 - \zeta_{n+1}).$$

Then we have the *stereographic projection*:

$$(2.28) \quad F = \hat{\pi} \circ \hat{F} : S^{4n+3} \longrightarrow \mathcal{H}^n, \quad (\zeta, \zeta_{n+1}) \longmapsto (q, t),$$

given by

$$q = (1 + \zeta_{n+1})^{-1}\zeta, \quad t = \frac{2\text{Im}\zeta_{n+1}}{|1 + \zeta_{n+1}|^2}.$$

Proposition 2.1. (cf. p.146 in [15]) *The Cayley transform is a conformal qc diffeomorphism between \mathcal{H}^n with its standard qc contact structure Θ_0 in (2.18) and the sphere minus a point with its standard qc contact structure Θ_S in (2.24). More precisely,*

$$(2.29) \quad F^*\Theta_0 = \alpha \frac{\Theta_S}{2|1 + \zeta_{n+1}|^2} \bar{\alpha},$$

where $\alpha = \frac{\overline{1 + \zeta_{n+1}}}{|1 + \zeta_{n+1}|}$ is a unit quaternion.

We have the following corollary.

Corollary 2.1. *Suppose that g_S is the standard qc metric on S^{4n+3} and g_0 is the standard qc metric on \mathcal{H}^n . We have*

$$(2.30) \quad F^*g_0 = \frac{g_S}{2|1 + \zeta_{n+1}|^2},$$

where F is the stereographic projection defined by (2.28).

Proof. Let $\mathbb{I} = (I_1, I_2, I_3)$ be the standard qc structure on the group \mathcal{H}^n and $\mathbb{I}' = (I'_1, I'_2, I'_3)$ be the standard qc structure on the sphere. Let $\omega_s := g_0(I_s \cdot, \cdot) = d\theta_s$, if we write $\Theta_0 = (\theta_1, \theta_2, \theta_3)$. Similarly, let $\omega'_s := g_S(I'_s \cdot, \cdot) = d\theta'_s$, if we write $\Theta_S = (\theta'_1, \theta'_2, \theta'_3)$. Consider the fundamental 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3,$$

on the horizontal subspace $H_{0;p}$ for a fixe point $p \in \mathcal{H}^n$. It is known that an element $g \in \text{GL}(4n, \mathbb{R})$ preserving Ω if and only if $g \in \text{Sp}(n)\text{Sp}(1)$ (cf. Lemma 9.1 in [31]), where $\text{Sp}(n) = \{A \in \text{O}_{g_0}(4n); AI_s = I_s A, s = 1, 2, 3\}$ and $\text{Sp}(1) = \{a_1 I_1 + a_2 I_2 + a_3 I_3; a_1^2 + a_2^2 + a_3^2 = 1\}$. Here $A \in \text{O}_{g_0}(4n)$ means that A is orthogonal with respect to g_0 on $H_{0;p}$. Similar result holds for Ω' .

Write $\alpha\Theta_S\bar{\alpha} = (\cdots, \sum_{k=1}^3 a_{jk}\theta'_k, \cdots)$. It follows from (2.29) and direct calculation that $F^*\omega_j = \lambda \sum_{k=1}^3 a_{jk}\omega'_k$ as 2-forms on the horizontal subspace, where $\lambda = \frac{1}{2|1 + \zeta_{n+1}|^2}$, and so

$$F^*\Omega = \lambda^2\Omega'.$$

Then A preserving Ω implies F^*A preserving Ω' . This is because

$$\begin{aligned} \Omega'(F^*A(X_1), \cdots, F^*A(X_4)) &= \lambda^{-2}F^*\Omega(F_*^{-1}[A(F_*X_1)], \cdots, F_*^{-1}[A(F_*X_4)]) \\ &= \lambda^{-2}\Omega(A(F_*X_1), \cdots, A(F_*X_4)) \\ &= \lambda^{-2}\Omega(F_*X_1, \cdots, F_*X_4) = \Omega'(X_1, \cdots, X_4), \end{aligned}$$

where $X_1, \cdots, X_4 \in H_{0;p}$. Thus F^*A is orthogonal with respect to g_S if A is orthogonal with respect to g_0 , and vise versa. But for any $X, Y \in H_{S,p}$ with $\|F_*X\|_{g_0} = \|F_*Y\|_{g_0}$, there exists an $A \in \text{O}_{g_0}(H)$, such that $F_*Y = AF_*X$. Then we have

$$\|Y\|_{g_S} = \|F_*^{-1}(F_*Y)\|_{g_S} = \|F_*^{-1}[A(F_*X)]\|_{g_S} = \|F^*A(X)\|_{g_S} = \|X\|_{g_S}.$$

This implies that F^*g_0 is conformal to g_S , i.e. we can write $F^*g_0 = \mu g_S$, for some $\mu > 0$. Consequently, we have $F^*\text{Vol}|_{H_0} = \mu^{2n} \text{Vol}|_{H_S}$.

On the other hand, $F^*d\theta_s|_{H_S} = \lambda \alpha d\theta'_s \bar{\alpha}|_{H_S} = \lambda g_S(\hat{I}_s \cdot, \cdot)|_{H_S} = \lambda \hat{\omega}_s|_{H_S}$ for $s = 1, 2, 3$, where $\hat{\mathbb{I}} = \alpha \mathbb{I}' \bar{\alpha}$, and \hat{I}_s is also an almost complex structure compatible with g_S . It follows that $(F^*d\theta_s)^{2n} = \lambda^{2n} \hat{\omega}_s^{2n}$ on H_S , and so $F^*\text{Vol}|_{H_0} = \lambda^{2n} \text{Vol}|_{H_S}$. So we must have $\mu = \lambda$. The corollary is proved. \square

Then by the transformation formula (2.30), we get the following corollary.

Corollary 2.2. *For any $\xi = (q, t) \in \mathcal{H}^n$, we have*

$$(2.31) \quad R^*g_0|_\xi = \frac{1}{\|\xi\|^4} g_0 \Big|_\xi, \quad D_r^*g_0 = r^2 g_0.$$

Proof. Note that the Cayley transformation \hat{F} maps $(-\zeta, -\zeta_{n+1})$ to

$$(2.32) \quad (-(1 - \zeta_{n+1})^{-1}\zeta, (1 - \zeta_{n+1})^{-1}(1 + \zeta_{n+1}))$$

for $(\zeta, \zeta_{n+1}) \in S^{4n+3}$. The reflection R in (2.12) on the Heisenberg group induces a reflection $\hat{R} := \hat{\pi}^{-1} \circ R \circ \hat{\pi}$ on the quadratic surface Σ in (2.26). It is direct to see that

$$\hat{F}(-\zeta, -\zeta_{n+1}) = (-q_{n+1}^{-1}q, q_{n+1}^{-1}) = \hat{R}(q, q_{n+1}),$$

for $(q, q_{n+1}) = \hat{F}(\zeta, \zeta_{n+1})$ by (2.27). Then $\hat{F} \circ \varphi \circ \hat{F}^{-1} = \hat{R}$, where φ is the qc isometric automorphism of S^{4n+3} given by $(\zeta, \zeta_{n+1}) \rightarrow (-\zeta, -\zeta_{n+1})$. Consequently, $F \circ \varphi \circ F^{-1} = R$. Then, by Corollary 2.1, we have

$$R^*g_0|_\xi = (F^{-1})^* \circ \varphi^* \circ F^*g_0 \Big|_\xi = \frac{|1 + \zeta_{n+1}|^2}{|1 - \zeta_{n+1}|^2} g_0 \Big|_\xi = \frac{1}{|q_{n+1}|^2} g_0|_\xi = \frac{1}{\|\xi\|^4} g_0 \Big|_\xi,$$

by (2.27), where $\xi = (\zeta, \zeta_{n+1})$. The result follows. \square

2.4. The conformal action of $\text{Sp}(n+1, 1)$ on the sphere. Let us show the elements of $\text{Sp}(n+1, 1)$ acting conformally on the sphere with the standard qc structure. We need to know the conformal factor explicitly.

Proposition 2.2. *Suppose that $\gamma \in \text{Sp}(n+1, 1)$ and Θ_S is the standard contact form on S^{4n+3} . Then we have*

$$(2.33) \quad \gamma^*\Theta_S|_\zeta = \lambda \frac{\Theta_S}{|[(\zeta, 1)\gamma]_{n+2}|^2} \bar{\lambda}$$

at point $\zeta \in S^{4n+3}$, where

$$(2.34) \quad \lambda = \frac{\overline{|[(\zeta, 1)\gamma]_{n+2}|}}{|[(\zeta, 1)\gamma]_{n+2}|}$$

is a unit quaternion.

Proof. Note that for any $x \in \mathbb{H}$,

$$(2.35) \quad d(x^{-1}) = -\bar{x} \frac{dx}{|x|^4} \bar{x}.$$

By the action of $\text{Sp}(n+1, 1)$ on the ball \bar{B}^{4n+4} in (2.21), we have

$$d[(\zeta, 1)\gamma]_l = \sum_{m=1}^{n+1} d\zeta_m \gamma_{ml} = [(d\zeta, 0)\gamma]_l,$$

for $\zeta \in \mathbb{H}^{n+1}$, $l = 1, \dots, n+1$, where $d\zeta = (d\zeta_1, \dots, d\zeta_{n+1})$. We differentiate (2.21) to get

$$d\gamma(\zeta) = \left(\dots, -\frac{\overline{[(\zeta, 1)\gamma]_{n+2}}[(d\zeta, 0)\gamma]_{n+2}\overline{[(\zeta, 1)\gamma]_{n+2}}}{|[(\zeta, 1)\gamma]_{n+2}|^4}[(\zeta, 1)\gamma]_l + [(\zeta, 1)\gamma]_{n+2}^{-1}[(d\zeta, 0)\gamma]_l, \dots \right).$$

We find that $\overline{\gamma(\zeta)_l} = \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{[(\zeta, 1)\gamma]_{n+2}^{-1}}$ and

$$\begin{aligned} (2.36) \quad & \sum_{l=1}^{n+1} d\gamma(\zeta)_l \overline{\gamma(\zeta)_l} \\ &= -\frac{\overline{[(\zeta, 1)\gamma]_{n+2}}[(d\zeta, 0)\gamma]_{n+2}\overline{[(\zeta, 1)\gamma]_{n+2}}}{|[(\zeta, 1)\gamma]_{n+2}|^4} \sum_{l=1}^{n+1} |[(\zeta, 1)\gamma]_l|^2 + [(\zeta, 1)\gamma]_{n+2}^{-1} \sum_{l=1}^{n+1} [(d\zeta, 0)\gamma]_l \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{[(\zeta, 1)\gamma]_{n+2}^{-1}} \\ &= -\frac{\overline{[(\zeta, 1)\gamma]_{n+2}}[(d\zeta, 0)\gamma]_{n+2}}{|[(\zeta, 1)\gamma]_{n+2}|^2} + [(\zeta, 1)\gamma]_{n+2}^{-1} \sum_{l=1}^{n+1} [(d\zeta, 0)\gamma]_l \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{[(\zeta, 1)\gamma]_{n+2}^{-1}} \\ &= \lambda \frac{\sum_{l=1}^{n+1} d\zeta_l \bar{\zeta}_l}{|[(\zeta, 1)\gamma]_{n+2}|^2} \bar{\lambda}, \end{aligned}$$

by γ preserving the hyperhermitian form $Q(\cdot, \cdot)$ in (1.1), i.e.,

$$(2.37) \quad -\sum_{l=1}^{n+1} [(\zeta, 1)\gamma]_l \overline{[(\zeta', 1)\gamma]_l} + [(\zeta, 1)\gamma]_{n+2} \overline{[(\zeta', 1)\gamma]_{n+2}} = -\zeta \bar{\zeta}' + 1$$

with $\zeta = \zeta'$ in the third identity and $|\zeta|^2 = 1$. The last identity follows from differentiating (2.37) with respect to ζ and setting $\zeta' = \zeta$ to get

$$\sum_{l=1}^{n+1} [(d\zeta, 0)\gamma]_l \overline{[(\zeta, 1)\gamma]_l} - [(d\zeta, 0)\gamma]_{n+2} \overline{[(\zeta, 1)\gamma]_{n+2}} = \sum_{l=1}^{n+1} d\zeta_l \bar{\zeta}_l.$$

Then (2.36) minus its conjugate gives

$$\sum_{l=1}^{n+2} \left(d\gamma(\zeta)_l \overline{\gamma(\zeta)_l} - \gamma(\zeta)_l \overline{d\gamma(\zeta)_l} \right) = \lambda \frac{\Theta_S}{|[(\zeta, 1)\gamma]_{n+2}|^2} \bar{\lambda}.$$

The proposition is proved. \square

Let X_0 be the tangential vector given by differentiating along a curve $\eta(t) = (\eta_1(t), \dots, \eta_{n+1}(t))$ at point ζ . Then the tangential vector $\gamma_* X_0$ at point $\gamma(\zeta)$ is given by differentiating along the curve $\gamma(\eta(t))$, i.e. $(\frac{d}{dt}|_{t=0} \gamma(\eta(t))_1, \dots, \frac{d}{dt}|_{t=0} \gamma(\eta(t))_{n+1})$, where

$$(2.38) \quad \frac{d}{dt} \Big|_{t=0} \gamma(\eta(t))_l = [(\zeta, 1)\gamma]_{n+2}^{-1}[(\eta'(0), 0)\gamma]_l - A_\zeta \cdot [(\zeta, 1)\gamma]_l$$

by the definition (2.21). Here

$$(2.39) \quad A_\zeta = \frac{d}{dt} \Big|_{t=0} [(\eta(t), 1)\gamma]_{n+2}^{-1} = \frac{\overline{[(\zeta, 1)\gamma]_{n+2}}[(\eta'(0), 0)\gamma]_{n+2}\overline{[(\zeta, 1)\gamma]_{n+2}}}{|[(\zeta, 1)\gamma]_{n+2}|^4} \in \mathbb{H}$$

by (2.35).

The following proposition gives us the conformal factor of the transformation of $\text{Sp}(n+1, 1)$ on the sphere with the standard qc metric.

Proposition 2.3. *For $\gamma \in \mathrm{Sp}(n+1, 1)$, we have*

$$(2.40) \quad H_{\gamma(\zeta)} = \gamma_* H_\zeta \quad \text{and} \quad \gamma^* g_S = \frac{1}{|[(\zeta, 1)\gamma]_{n+2}|^2} g_S.$$

Proof. Let η be a curve in S^{4n+3} with $\eta(0) = \zeta$ such that $\eta'(0)$ is horizontal. By γ preserving the hyperhermitian form in (2.37) again, we have

$$(2.41) \quad - \sum_{l=1}^{n+1} [(\eta(t), 1)\gamma]_l \overline{[(\eta(s), 1)\gamma]_l} + [(\eta(t), 1)\gamma]_{n+2} \overline{[(\eta(s), 1)\gamma]_{n+2}} = -\langle \eta(t), \eta(s) \rangle + 1.$$

Differentiate it with respect to t at 0 and then let $s \rightarrow 0$ to get

$$(2.42) \quad - \sum_{l=1}^{n+1} [(\eta'(0), 0)\gamma]_l \overline{[(\zeta, 1)\gamma]_l} + [(\eta'(0), 0)\gamma]_{n+2} \overline{[(\zeta, 1)\gamma]_{n+2}} = -\langle \eta'(0), \zeta \rangle = 0$$

by $\eta'(0) \in H_\zeta$. Then we have

$$\left\langle \frac{d\gamma(\eta(t))}{dt} \Big|_{t=0}, \gamma(\zeta) \right\rangle = \sum_{l=1}^{n+1} ([(\zeta, 1)\gamma]_{n+2}^{-1} \cdot [(\eta'(0), 0)\gamma]_l - A_\zeta \cdot [(\zeta, 1)\gamma]_l) \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{[(\zeta, 1)\gamma]_{n+2}^{-1}} = 0,$$

by (2.38), (2.42) and

$$(2.43) \quad A_\zeta \sum_{l=1}^{n+1} [(\zeta, 1)\gamma]_l \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{[(\zeta, 1)\gamma]_{n+2}^{-1}} = A_\zeta \cdot [(\zeta, 1)\gamma]_{n+2} = [(\zeta, 1)\gamma]_{n+2}^{-1} \cdot [(\eta'(0), 0)\gamma]_{n+2}$$

by (2.37) and (2.39). It implies that $\frac{d}{dt} \Big|_{t=0} \gamma(\eta(t)) \in H_{\gamma(\zeta)}$. Thus $\gamma_* H_\zeta \subset H_{\gamma(\zeta)}$. They actually coincide since γ is invertible. Denote $\|X\|^2 := g_S(X, X)$ for $X \in H$. Moreover, by (2.38) and (2.42) again we have

$$\begin{aligned} \frac{1}{4} \|\gamma_* X_0\|^2 &= \sum_{l=1}^{n+1} \left| \frac{d}{dt} \Big|_{t=0} \gamma(\eta(t))_l \right|^2 = \sum_{l=1}^{n+1} |[(\zeta, 1)\gamma]_{n+2}^{-1} [(\eta'(0), 0)\gamma]_l|^2 \\ &\quad + \sum_{l=1}^{n+1} |A_\zeta [(\zeta, 1)\gamma]_l|^2 - 2 \operatorname{Re} \sum_{l=1}^{n+1} [(\zeta, 1)\gamma]_{n+2}^{-1} [(\eta'(0), 0)\gamma]_l \cdot \overline{[(\zeta, 1)\gamma]_l} \cdot \overline{A_\zeta} \\ &= \frac{\sum_{l=1}^{n+1} |[(\eta'(0), 0)\gamma]_l|^2}{|[(\zeta, 1)\gamma]_{n+2}|^2} + |A_\zeta|^2 \sum_{l=1}^{n+1} |[(\zeta, 1)\gamma]_l|^2 - 2 \frac{|[(\eta'(0), 0)\gamma]_{n+2}|^2}{|[(\zeta, 1)\gamma]_{n+2}|^2} = \frac{\sum_{l=1}^{n+1} |\eta'_l(0)|^2}{|[(\zeta, 1)\gamma]_{n+2}|^2}. \end{aligned}$$

Here we have used

$$\sum_{l=1}^{n+1} |[(\eta'(0), 0)\gamma]_l|^2 - |[(\eta'(0), 0)\gamma]_{n+2}|^2 = \sum_{l=1}^{n+1} |\eta'_l(0)|^2,$$

which follows from differentiating (2.41) with respect to s and t at 0. Then by $\gamma^* g_S(X_0, X_0) = g_S(\gamma_* X_0, \gamma_* X_0)$, we complete the proof of (2.40). \square

Now we give the pull back formula of the quaternionic structure \mathbb{I} in the following proposition.

Corollary 2.3. *For $\gamma \in \mathrm{Sp}(n+1, 1)$, we have*

$$\gamma^* \mathbb{I} = \lambda \mathbb{I} \bar{\lambda},$$

where λ is a unit quaternion given by (2.34).

Proof. It follows from (2.4) in the proof of Proposition 2.2. \square

2.5. Spherical qc manifolds and connected sums. A local sphere theorem was proved by Ivanov and Vassilev in [13], i.e. a qc manifold is spherical if and only if its conformal qc curvature of the Biquard connection vanishes. See also [1] for a proof using parabolic geometry. The following proposition tells us that this definition coincides with the topological definition gives at the beginning of this paper.

Proposition 2.4. *A qc manifold (M, g, \mathbb{Q}) is spherical if and only if it is a manifold with coordinate charts $\{(U_i, \phi_i)\}$, where $\phi_i : U_i \rightarrow S^{4n+3}$ and transition maps are given by induced action of elements of $\mathrm{Sp}(n+1, 1)$.*

Proof. The necessity follows from the qc Liouville type Theorem 2.2.

Given such coordinate charts $\{(U_i, \phi_i)\}$ of M , let us construct a qc metric g and a bundle \mathbb{Q} on M . Let χ_i be a unit partition subordinating to the cover $\{U_i\}$, i.e. $\mathrm{supp}\chi_i \subset U_i$ and $\sum_i \chi_i \equiv 1$. Let $g := \sum_k \chi_k \phi_k^* g_S$ be a Carnot-Carathéodory metric on $H_p = \phi_i^* H_{\phi_i(p)}$, where $H_{\phi_i(p)}$ is the horizontal subspace of the sphere at the point $\phi_i(p)$. This definition of H_p is independent of the choice of i , since $(\phi_{ji})_* H_{\phi_i(p)} = H_{\phi_j(p)}$ by Proposition 2.3, where transition map $\phi_{ji} := \phi_j \circ \phi_i^{-1} \in \mathrm{Sp}(n+1, 1)$. Then on the open set $\phi_i(U_i) \subset S^{4n+3}$, we have

$$(\phi_i^{-1})^* g = (\phi_i^{-1})^* \left(\sum_k \chi_k \phi_k^* g_S \right) = \chi_i \circ \phi_i^{-1} \cdot g_S + \sum_{k \neq i} \chi_k \circ \phi_i^{-1} \cdot \phi_{ki}^* g_S = B g_S$$

for some positive function B on $\phi_i(U_i)$ by the pull back formula (2.40) for the metric g_S in Proposition 2.3. Therefore g is a spherical qc metric on M . By Proposition 2.3, we have

$$\phi_{ji}^* \mathbb{I} = \nu \mathbb{I} \bar{\nu} \quad \text{with } \nu(\zeta) = \frac{[(\zeta, 1)\phi_{ji}]_{n+2}}{|[(\zeta, 1)\phi_{ji}]_{n+2}|},$$

on $U_i \cap U_j$. We have $\phi_{ii}^*(\mathbb{I}) = \mathbb{I}$ and $\phi_{kj}^* \circ \phi_{ji}^*(\mathbb{I}) = \phi_{ki}^*(\mathbb{I})$. Namely ϕ_{ji}^* satisfy the cocycle condition and define a sphere bundle. We can choose $\Theta = \phi_i^*(B\Theta_S)$ locally, and the compatibility condition (2.2)-(2.3) for g and Θ obviously holds, since $d\Theta = \phi_i^*(Bd\Theta_S)$ when restricted to the horizontal subspace. \square

For $\xi \in \mathcal{H}^n$ and $\epsilon > 0$, define a *ball on the quaternionic Heisenberg group* as

$$B_{\mathcal{H}}(\xi, \epsilon) := \{\eta \in \mathcal{H}^n; \|\xi^{-1} \cdot \eta\| < \epsilon\}.$$

Let (M, g, \mathbb{Q}) be a spherical qc manifold of dimension $4n+3$ with two punctures η_1, η_2 , or disjoint union of two connected spherical qc manifolds $(M_{(1)}, g_{(1)}, \mathbb{Q}_{(1)})$, $(M_{(2)}, g_{(2)}, \mathbb{Q}_{(2)})$ with one puncture $\eta_i \in M_{(i)}$, $i = 1, 2$, each. Let U_1 and U_2 be two disjoint neighborhoods of η_1 and η_2 , respectively. Let

$$(2.44) \quad \psi_i : U_i \rightarrow B_{\mathcal{H}}(0, 2), \quad i = 1, 2,$$

be local coordinate charts such that $\psi_i(\eta_i) = 0$. For $t < 1$, define

$$\begin{aligned} U_i(t, 1) &:= \{\eta \in U_i; t < \|\psi_i(\eta)\| < 1\}, \\ U_i(t) &:= \{\eta \in U_i; \|\psi_i(\eta)\| < t\}, \end{aligned}$$

$i = 1, 2$. For any $t \in (0, 1)$, $A \in \mathrm{Sp}(n)$ and $\sigma \in \mathrm{Sp}(1)$, we can form a new spherical qc manifold $M_{t,\sigma,A}$ by removing the closed balls $\overline{U_i(t)}$, $i = 1, 2$, and gluing $U_1(t, 1)$ with $U_2(t, 1)$ by the conformal qc mapping $\Psi_{t,\sigma,A} : U_1(t, 1) \rightarrow U_2(t, 1)$ defined by

$$(2.45) \quad \Psi_{t,\sigma,A}(\eta) = \psi_2^{-1} \circ D_t \circ R \circ \sigma \circ A \circ \psi_1(\eta), \text{ for } \eta \in U_1(t, 1),$$

where $R : \{\zeta \in \mathcal{H}^n; t < \|\zeta\| < 1\} \rightarrow \{\zeta \in \mathcal{H}^n; 1 < \|\zeta\| < \frac{1}{t}\}$ is the inversion in (2.12). Note that $\Psi_{t,\sigma,A}$ is conformal qc with respect to the spherical qc structure \mathbb{Q} on M , since $D_t \circ R \circ \sigma \circ U_A$ is a qc automorphism of the quaternionic Heisenberg group. Such a conformal qc transformation maps $U_1(t, 1)$ to $U_2(t, 1)$, which identifies the inner boundary of $U_1(t, 1)$ with the outer boundary $U_2(t, 1)$ and vice versa. Let

$$(2.46) \quad \pi_{t,\sigma,A} : (M_1 \setminus \overline{U_1(t)}) \cup (M_2 \setminus \overline{U_2(t)}) \rightarrow M_{t,\sigma,A}$$

be a canonical projection. We call $M_{t,\sigma,A}$ the *connected sum* of M_1 and M_2 , which is a spherical qc manifold by Proposition 2.4. We denote this spherical qc manifold by $(M_{t,\sigma,A}, g, \mathbb{Q}_{t,\sigma,A})$, where g is a metric in the conformal class given by Proposition 2.4. As in the locally conformally case, the connected sums are expected to be not isomorphic for some different choices of t, σ, A [19]. As in the locally conformally flat case, it is interesting to investigate their moduli space.

3. THE QC YAMABE OPERATOR AND ITS GREEN FUNCTION

3.1. The qc Yamabe operator. On a qc manifold (M, g, \mathbb{Q}) , let us choose a local basis $\{X_j\}_{j=1}^{4n}$ of the horizontal subspace H as in [36]. We choose a local section e_1 of H such that $g(e_1, e_1) = 1$. Then, $e_1, I_1 e_1, I_2 e_1, I_3 e_1$ are mutually orthonormal. Set $I_0 = \mathrm{id}_H$. Now choose a local section e_2 of H orthonormal to $\mathrm{span}\{I_k e_1 | k = 0, \dots, 3\}$. Then $e_2, I_1 e_2, I_2 e_2, I_3 e_2$ are mutually orthonormal again and $\mathrm{span}\{I_k e_1 | k = 0, \dots, 3\} \perp \mathrm{span}\{I_k e_2 | k = 0, \dots, 3\}$. Repeating this procedure, we can find e_1, \dots, e_n such that $\{I_k e_j | j = 1, \dots, n, k = 0, \dots, 3\}$ is a local orthonormal basis of H under the metric g . Set

$$X_{4l+\alpha+1} := \sqrt{2} I_\alpha e_{l+1}$$

for $l = 0, \dots, n-1$, $\alpha = 0, \dots, 3$. Then

$$g(X_j, X_k) = 2\delta_{jk}.$$

The Carnot-Carathéodory metric g induces a dual metric on H^* , denoted by $\langle \cdot, \cdot \rangle_g$. Since the Biquard connection ∇ preserves H , we can write the covariant derivative as $\nabla_{X_j} X_k = \Gamma_{jk}^{k'} X_{k'}$. By definition, $(\nabla_X \omega)Y = \nabla_X(\omega(Y)) - \omega(\nabla_X Y)$ for a 1-form $\omega \in \Omega^1(M)$. Then we define an L^2 inner product $\langle \cdot, \cdot \rangle_{g,\mathbb{Q}}$ on $\Gamma(H^*)$ by

$$\langle \omega, \omega' \rangle_{g,\mathbb{Q}} := \int_M \langle \omega, \omega' \rangle_g dV_{g,\mathbb{Q}},$$

where the volume form $dV_{g,\mathbb{Q}}$ is

$$(3.1) \quad dV_{g,\mathbb{Q}} := \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge (d\theta_s)^{2n},$$

$s = 1, 2, 3$, if we choose $\Theta = (\theta_1, \theta_2, \theta_3)$ locally. The volume form is independent of s and the choice of Θ by the follow proposition.

Proposition 3.1. *The volume element $dV_{g,\mathbb{Q}}$ only depends on g and \mathbb{Q} , not on s or the choice of the \mathbb{R}^3 -valued contact form $\Theta = (\theta_1, \theta_2, \theta_3)$.*

Proof. Let 1-forms $\{\theta^j\}_{j=1}^{4n}$ be the basis dual to $\{X_j\}_{j=1}^{4n}$. Recall the structure equation (3.6) in [36], i.e.

$$d\theta_s = - \sum_{l=0}^{n-1} \sum_{j,k=1}^4 b_{kj}^s \theta^{4l+k} \wedge \theta^{4l+j}, \quad \text{mod } \theta_1, \theta_2, \theta_3,$$

$s = 1, 2, 3$, where b^s is given by (2.15). By using b^1 in (2.15), it is direct to check that

$$(d\theta_1)^{2n} = \left(\sum_{l=0}^{n-1} 2\theta^{4l+1} \wedge \theta^{4l+2} + 2\theta^{4l+3} \wedge \theta^{4l+4} \right)^{2n} = 2^{2n}(2n)! \theta^1 \wedge \cdots \wedge \theta^{4n}, \quad \text{mod } \theta_1, \theta_2, \theta_3.$$

Similarly,

$$(3.2) \quad (d\theta_s)^{2n} = 2^{2n}(2n)! \theta^1 \wedge \cdots \wedge \theta^{4n}, \quad \text{mod } \theta_1, \theta_2, \theta_3,$$

$s = 2, 3$, and so $dV_{g,\mathbb{Q}}$ is independent of s . Let $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ be another contact form satisfying $d\tilde{\theta}_s(X, Y) = g(I_s X, Y)$. We can write $\tilde{\theta}_s = \sum_{j=1}^3 c_{sj} \theta_j$, $s = 1, 2, 3$, for some $\text{SO}(3)$ -valued function (c_{ij}) by Lemma 2.1 and simultaneously $\tilde{I}_s = \sum_{j=1}^3 c_{sj} I_j$, $s = 1, 2, 3$, by the proof of Lemma 2.1. As the procedure at the beginning of this section, we can choose orthonormal basis $\{\cdots, e_k, \tilde{I}_1 e_k, \tilde{I}_2 e_k, \tilde{I}_3 e_k, \cdots\}$. Then we have

$$\tilde{X}_{4l+1} = X_{4l+1}, \quad \tilde{X}_{4l+\alpha+1} = \sqrt{2} \tilde{I}_\alpha e_{l+1} = \sum_{\beta=1}^3 c_{\alpha\beta} X_{4l+\beta+1}$$

for $l = 0, \dots, n-1$, $\alpha = 1, 2, 3$. Dually, we have the dual basis $\{\tilde{\theta}^i\}$ such that

$$\tilde{\theta}^{4l+1} = \theta^{4l+1}, \quad \tilde{\theta}^{4l+\alpha+1} = \sum_{\beta=1}^3 c_{\alpha\beta} \theta^{4l+\beta+1},$$

for $l = 0, \dots, n-1$, $\alpha = 1, 2, 3$. In fact, we have $\tilde{\theta}^{4l+1} \wedge \cdots \wedge \tilde{\theta}^{4l+4} = \det(c_{ij}) \theta^{4l+1} \wedge \cdots \wedge \theta^{4l+4}$ and $\det(c_{ij}) = 1$ by $(c_{ij}) \in \text{SO}(3)$. By (3.2), we have

$$(3.3) \quad \begin{aligned} dV_{\tilde{g},\mathbb{Q}} &= \tilde{\theta}_1 \wedge \tilde{\theta}_2 \wedge \tilde{\theta}_3 \wedge \left(d\tilde{\theta}_s \right)^{2n} = 2^{2n}(2n)! \det(c_{ij}) \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \tilde{\theta}^1 \wedge \cdots \wedge \tilde{\theta}^{4n} \\ &= 2^{2n}(2n)! \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta^1 \wedge \cdots \wedge \theta^{4n} = \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge (d\theta_s)^{2n}. \end{aligned}$$

The proposition is proved. \square

Denote $d_b := \text{pr} \circ d$, where pr is the projection from T^*M to H^* . We define the *SubLaplacian* $\Delta_{g,\mathbb{Q}}$ associated to the qc contact structure (M, g, \mathbb{Q}) by

$$(3.4) \quad \int_M \Delta_{g,\mathbb{Q}} u \cdot v dV_{g,\mathbb{Q}} = \int_M \langle d_b u, d_b v \rangle_g dV_{g,\mathbb{Q}}$$

for $u, v \in C_0^\infty(M)$. The SubLaplacian $\Delta_{g,\mathbb{Q}}$ has the following expression.

Proposition 3.2. (cf. p.365 in [36]) *For $u \in C^\infty(M)$, we have*

$$(3.5) \quad \Delta_{g,\mathbb{Q}} u = \frac{1}{2} \sum_{j=1}^{4n} \left(-X_j X_j u + \sum_{k=1}^{4n} \Gamma_{kk}^j X_j u \right).$$

The transformation law of the scalar curvature under the conformal changes was given by Biquard in [3] for $\dim M > 7$. When $\dim M = 7$, it was given by Duchemin in [7].

Theorem 3.1. *Under the conformal change $\tilde{g} = f^2g$, the scalar curvature becomes*

$$s_{\tilde{g},\mathbb{Q}} = f^{-2} (s_{g,\mathbb{Q}} - 8(n+2)Tr^H \nabla \alpha - 16(n+1)(n+2)|\alpha|^2)$$

where $\alpha = f^{-1}df$ and ∇ is the Biquard connection.

Writing the conformal factor f as e^h , we can write the transformation law in the following form.

Corollary 3.1. *The scalar curvature $s_{\tilde{g},\mathbb{Q}}$ associated with the structure $\tilde{g} = e^{2h}g$ satisfies*

$$s_{\tilde{g},\mathbb{Q}} = e^{-2h} \left(s_{g,\mathbb{Q}} + 2(Q+2)\Delta_{g,\mathbb{Q}}h - \sum_{j=1}^{4n} \frac{(Q+2)(Q-2)}{2} (X_j h)^2 \right)$$

Proof. Since $f = e^h$, $\alpha = dh$. Then,

$$\begin{aligned} Tr^H \nabla \alpha &= \sum_{j=1}^{4n} (\nabla_{X_j/\sqrt{2}} \alpha)(X_j/\sqrt{2}) = \frac{1}{2} \sum_j^{4n} (\nabla_{X_j} dh)(X_j) \\ &= \frac{1}{2} \sum_j^{4n} (X_j(dh(X_j)) - dh(\nabla_{X_j} X_j)) = \frac{1}{2} \sum_{j=1}^{4n} \left(X_j X_j h - \sum_{k=1}^{4n} \Gamma_{kk}^j X_j h \right) = -\Delta_{g,\mathbb{Q}} h. \end{aligned}$$

The result follows. \square

Corollary 3.2. (cf. p. 360 in [36]) *The scalar curvature $s_{\tilde{g},\mathbb{Q}}$ associated with the structure $(M, \tilde{g}, \mathbb{Q})$ with $\tilde{g} = \phi^{\frac{4}{Q-2}}g$ satisfies qc Yamabe equation:*

$$(3.6) \quad b_n \Delta_{g,\mathbb{Q}} \phi + s_{g,\mathbb{Q}} \phi = s_{\tilde{g},\mathbb{Q}} \phi^{\frac{Q+2}{Q-2}}, \quad b_n = 4 \frac{Q+2}{Q-2}.$$

Now let us derive the transformation law of the qc Yamabe operator. See [28] for such derivation in the pseudoRiemannian case and [34] in the CR case.

Proposition 3.3. *Let $(M, \tilde{g}, \mathbb{Q})$ and (M, g, \mathbb{Q}) be two qc manifolds. Let $\tilde{g} = \phi^{\frac{4}{Q-2}}g$ for some positive smooth function ϕ on M . Then*

$$(3.7) \quad \Delta_{g,\mathbb{Q}}(\phi \cdot f) = \Delta_{g,\mathbb{Q}} \phi \cdot f + \phi^{\frac{Q+2}{Q-2}} \Delta_{\tilde{g},\mathbb{Q}} f$$

for any smooth function f on M .

Proof. Let $\Theta = (\theta_1, \theta_2, \theta_3)$ be a \mathbb{R}^3 -valued 1-form associated to (M, g, \mathbb{Q}) and let $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ be associated to $(M, \tilde{g}, \mathbb{Q})$. For any real function h on M , we have

$$\begin{aligned} \langle \Delta_{g,\mathbb{Q}}(\phi \cdot f), h \rangle_{g,\mathbb{Q}} &= \int_M \langle d_b(\phi \cdot f), d_b h \rangle_g dV_{g,\mathbb{Q}} = \langle d_b \phi \cdot f + \phi d_b f, d_b h \rangle_{g,\mathbb{Q}} \\ (3.8) \quad &= \langle d_b \phi, f \cdot d_b h \rangle_{g,\mathbb{Q}} + \langle d_b f, \phi \cdot d_b h \rangle_{g,\mathbb{Q}} \\ &= \langle d_b \phi, d_b(f \cdot h) \rangle_{g,\mathbb{Q}} + \langle d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{g,\mathbb{Q}} \\ &= \langle \Delta_{g,\mathbb{Q}} \phi, f \cdot h \rangle_{g,\mathbb{Q}} + \langle d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{g,\mathbb{Q}}. \end{aligned}$$

Let us calculate the second term in the right side of (3.8). By our assumption and Proposition 3.1, we just need to consider $\tilde{\Theta} = \phi^{\frac{4}{Q-2}}\Theta$. Then for a fixed s , we have

$$(3.9) \quad d\tilde{\theta}_s = d(\phi^{\frac{4}{Q-2}}\theta_s) = \frac{4}{Q-2}\phi^{\frac{6-Q}{Q-2}}d\phi \wedge \theta_s + \phi^{\frac{4}{Q-2}}d\theta_s,$$

i.e. $d\tilde{\theta}_s = \phi^{\frac{4}{Q-2}}d\theta_s \bmod \theta_1, \theta_2, \theta_3$. So, we get

$$(3.10) \quad dV_{\tilde{g},\mathbb{Q}} = \tilde{\theta}_s \wedge \tilde{\theta}_2 \wedge \tilde{\theta}_3 \wedge (d\tilde{\theta}_s)^{2n} = \phi^{\frac{2Q}{Q-2}}\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge (d\theta_1)^{2n} = \phi^{\frac{2Q}{Q-2}}dV_{g,\mathbb{Q}}.$$

Consequently, for 1-forms $\omega_1, \omega_2 \in \Gamma(H^*)$, we have

$$\langle \omega_1, \omega_2 \rangle_{g,\mathbb{Q}} = \int_M \langle \omega_1, \omega_2 \rangle_g dV_{g,\mathbb{Q}} = \langle \phi^{-2}\omega_1, \omega_2 \rangle_{\tilde{g},\mathbb{Q}}.$$

Now we find that

$$\begin{aligned} \langle d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{g,\mathbb{Q}} &= \langle \phi^{-2}d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{\tilde{g},\mathbb{Q}} = \langle d_b f, d_b(\phi^{-1}h) \rangle_{\tilde{g},\mathbb{Q}} \\ &= \int \Delta_{\tilde{g},\mathbb{Q}} f \cdot \phi^{-1}h dV_{\tilde{g},\mathbb{Q}} = \int \phi^{\frac{Q+2}{Q-2}} \Delta_{\tilde{g},\mathbb{Q}} f \cdot h dV_{g,\mathbb{Q}}. \end{aligned}$$

The proposition is proved. \square

Corollary 3.3. *The qc Yamabe operator $L_{g,\mathbb{Q}}$ satisfies the transformation law*

$$(3.11) \quad L_{\tilde{g},\mathbb{Q}} f = \phi^{-\frac{Q+2}{Q-2}} L_{g,\mathbb{Q}}(\phi f),$$

if $\tilde{g} = \phi^{\frac{4}{Q-2}}g$ and $f \in C^\infty(M)$.

Proof. By using (3.6) and (3.7), we have

$$\begin{aligned} L_{g,\mathbb{Q}}(\phi f) &= b_n \Delta_{g,\mathbb{Q}}(\phi f) + s_{g,\mathbb{Q}} \phi f = b_n \left(\Delta_{g,\mathbb{Q}} \phi \cdot f + \phi^{\frac{Q+2}{Q-2}} \Delta_{\tilde{g},\mathbb{Q}} f \right) + s_{g,\mathbb{Q}} \phi f \\ &= \phi^{\frac{Q+2}{Q-2}} (b_n \Delta_{\tilde{g},\mathbb{Q}} f + s_{\tilde{g},\mathbb{Q}} f). \end{aligned}$$

The result follows. \square

3.2. The Green function of the qc Yamabe operator. For simplicity we write $G_{g_0,\mathbb{Q}}, \Delta_{g_0,\mathbb{Q}}, L_{g_0,\mathbb{Q}}, \langle \cdot, \cdot \rangle_{g_0,\mathbb{Q}}$ as $G_0, \Delta_0, L_0, \langle \cdot, \cdot \rangle_0$. Since the Biquard connection on \mathcal{H}^n is trivial [15], its connections and curvatures vanish. For $u \in C^1(\mathcal{H}^n)$, we have

$$du = \sum_{j=1}^{4n} Y_j u \cdot \theta^j + \sum_{s=1}^3 \frac{\partial u}{\partial t_s} \cdot \theta_{0,s}$$

and $d_b u = \sum_{j=1}^{4n} Y_j u \cdot \theta^j$, where $\theta^j = dy_j$, and Y_j is given by (2.17). Recall that $\langle Y_j, Y_k \rangle_0 = 2\delta_{jk}$ and $\langle \theta^j, \theta^k \rangle_0 = \frac{1}{2}\delta_{jk}$ for $j, k = 1, \dots, 4n$. Hence,

$$\langle d_b u, d_b v \rangle_0 = \frac{1}{2} \sum_{j=1}^{4n} Y_j u \cdot Y_j v$$

for real functions u, v , and the SubLaplacian is

$$\Delta_0 = -\frac{1}{2} \sum_{j=1}^{4n} Y_j Y_j$$

by (3.5).

A continuous function $G_{g,\mathbb{Q}} : M \times M \setminus \text{diag} M \rightarrow \mathbb{R}$ is called the *Green function* of the qc Yamabe operator $L_{g,\mathbb{Q}}$ if

$$\int_M G_{g,\mathbb{Q}}(\xi, \eta) L_{g,\mathbb{Q}} u(\eta) dV_{g,\mathbb{Q}}(\eta) = u(\xi)$$

for all $u \in C_0^\infty(M)$. Namely, $L_{g,\mathbb{Q}} G_{g,\mathbb{Q}}(\xi, \cdot) = \delta_\xi$.

In [22] Kaplan found the explicit form of the fundamental solution of the SubLaplacian on groups of H-type. Since the quaternionic Heisenberg groups is a group of H-type, we know its fundamental solution.

Proposition 3.4. *The Green function of the qc Yamabe operator $L_0 = b_n \Delta_0$ on the quaternionic Heisenberg group \mathcal{H}^n with the pole at ξ is*

$$G_0(\xi, \eta) := \frac{C_Q}{\|\xi^{-1}\eta\|^{Q-2}},$$

for $\xi \neq \eta$, $\xi, \eta \in \mathcal{H}^n$, where $\|\cdot\|$ is the norm on \mathcal{H}^n defined by (2.7) and

$$(3.12) \quad C_Q^{-1} = 8(n+1)(n+2)b_n \int_{\mathbb{R}^{4n+3}} \frac{|y|^2}{(|y|^4 + |t|^2 + 1)^{n+3}} dV_0,$$

where dV_0 is Lebesgue measure.

In the Appendix, we prove this proposition by simple and direct calculation.

Proposition 3.5. *Let (M, g, \mathbb{Q}) be a connected compact qc manifold. Then one and only one of the following cases holds: there is a qc metric \tilde{g} conformal to g which have either positive, negative or vanishing scalar curvature everywhere.*

Proof. The qc Yamabe operator $L_{g,\mathbb{Q}}$ is a formally self-adjoint and subelliptic differential operator. So its spectrum is real and bounded from below. Let λ_1 be the first eigenvalue of $L_{g,\mathbb{Q}}$ and let ϕ be an eigenfunction of $L_{g,\mathbb{Q}}$ with eigenvalue λ_1 . Then $\phi > 0$ and is C^∞ by Theorem 3.6 in [17]. The scalar curvature of $(M, \tilde{g}, \mathbb{Q})$ with $\tilde{g} = \phi^{\frac{4}{Q-2}} g$ is $s_{\tilde{g},\mathbb{Q}} = \lambda_1 \phi^{-\frac{4}{Q-2}}$ by the qc Yamabe equation (3.6). In particular, $s_{\tilde{g},\mathbb{Q}} > 0$ (resp. $s_{\tilde{g},\mathbb{Q}} < 0$, resp. $s_{\tilde{g},\mathbb{Q}} \equiv 0$) if $\lambda_1 > 0$ (resp. $\lambda_1 < 0$, resp. $\lambda_1 = 0$). On the other hand, if \hat{g} has scalar curvature $s_{\hat{g},\mathbb{Q}} > 0$ (resp. $s_{\hat{g},\mathbb{Q}} < 0$, resp. $s_{\hat{g},\mathbb{Q}} \equiv 0$), the first eigenvalue $\hat{\lambda}_1$ of $L_{\hat{g},\mathbb{Q}}$ obviously satisfies $\hat{\lambda}_1 > 0$ (resp. $\hat{\lambda}_1 < 0$, resp. $\hat{\lambda}_1 = 0$). \square

Remark 3.1. *This proposition does not exclude the possibility that a scalar positive qc manifold has a metric with scalar curvature vanishing or negative somewhere.*

Define the qc Yamabe invariant

$$(3.13) \quad \lambda(M, g, \mathbb{Q}) := \inf_{u>0} \frac{\int_M (b_n |\nabla_g u|^2 + s_{g,\mathbb{Q}} u^2) dV_{g,\mathbb{Q}}}{\left(\int_M u^{\frac{2Q}{Q-2}} dV_{g,\mathbb{Q}} \right)^{\frac{Q-2}{Q}}},$$

where $|\nabla_g f|^2 = \sum_{j=1}^{4n} |X_j f|^2$ if $\{X_j\}$ is a local orthogonal basis of H under the Carnot-Carathéodory metric g . It is an invariant for the conformal class of qc manifolds (cf. p. 361 in [36]).

Theorem 1.1 in [36] tells us that, if $\lambda(M, g, \mathbb{Q}) < \lambda(\mathcal{H}^n, g_0, \mathbb{Q}_0)$, then (3.13) can be achieved by a positive C^∞ solution u of the qc Yamabe equation (3.6), i.e. it has a constant scalar

curvature $s_{\tilde{g},\mathbb{Q}} = \lambda(M, g, \mathbb{Q})$ for $\tilde{g} = u^{\frac{4}{Q-2}}g$. It is known that $\lambda(\mathcal{H}^n, g_0, \mathbb{Q}_0) > 0$ (cf. Corollary 2.1 [18]). Thus, (M, g, \mathbb{Q}) is scalar negative or zero if and only if $\lambda(M, g, \mathbb{Q})$ is negative or zero, respectively. Consequently, (M, g, \mathbb{Q}) is scalar positive if and only if $\lambda(M, g, \mathbb{Q})$ is positive.

From now on in this section, we assume (M, g, \mathbb{Q}) to be connected, compact and scalar positive. In this case, the qc Yamabe operator $L_{g,\mathbb{Q}} = b_n \Delta_{g,\mathbb{Q}} + s_{g,\mathbb{Q}}$ is a positive operator and its inverse always exists. The Green function is the Schwarz kernel of the inverse operator and can be constructed as in the following proposition. Moreover, we also find the singular part of the Green function. In this case, the Green function of the qc Yamabe operator $L_{g,\mathbb{Q}}$ is unique.

Proposition 3.6. *Let (M, g, \mathbb{Q}) be a connected compact spherical qc manifold with positive scalar curvature and let U be a sufficiently small open set. Then the function $G_{g,\mathbb{Q}}(\xi, \eta) - \rho_{g,\mathbb{Q}}(\xi, \eta)$ can be extended to a C^∞ function on $U \times U$, where $\rho_{g,\mathbb{Q}}(\cdot, \cdot)$ is given by (1.3).*

Proof. The proof is similar to the CR case. Suppose that $\bar{U} \subset \tilde{U} \subset \mathcal{H}^n$ and $g = \phi^{\frac{4}{Q-2}}g_0$ on \tilde{U} . We choose a sufficiently small ρ such that $B_{\mathcal{H}}(\xi, \rho) \subset \tilde{U}$ for any $\xi \in U$. We can construct the Green function as follows. For $\xi, \eta \in U$, define

$$\tilde{G}(\xi, \eta) = \tilde{G}(\xi^{-1}\eta),$$

where \tilde{G} is the cut-off fundamental solution, i.e. $\tilde{G}(\tilde{\eta}) = \frac{C_Q}{\|\tilde{\eta}\|^{Q-2}}f(\tilde{\eta})$ for $\tilde{\eta} \in \mathcal{H}^n$. Here $f \in C_0^\infty(\mathcal{H}^n)$ satisfying $f \equiv 1$ on $B_{\mathcal{H}}(0, \frac{\rho}{2})$ and $f \equiv 0$ on $B_{\mathcal{H}}(0, \rho)^c$. Recall that $L_0 = -\frac{b_n}{2} \sum_{j=1}^{4n} Y_j Y_j$, where Y_j are given by (2.17). Then,

$$\begin{aligned} (3.14) \quad L_0 \tilde{G}(\tilde{\eta}) &= L_0 \left(\frac{C_Q}{\|\tilde{\eta}\|^{Q-2}} f(\tilde{\eta}) \right) = \delta_0 - b_n \sum_{j=1}^{4n} Y_j \left(\frac{C_Q}{\|\tilde{\eta}\|^{Q-2}} \right) Y_j f(\tilde{\eta}) + \frac{C_Q}{\|\tilde{\eta}\|^{Q-2}} L_0 f(\tilde{\eta}) \\ &=: \delta_0 + \tilde{G}_1(\tilde{\eta}) \end{aligned}$$

by $\frac{C_Q}{\|\tilde{\eta}\|^{Q-2}}$ being the fundamental solution of L_0 and $Y_j f \equiv 0$ on $B_{\mathcal{H}}(0, \frac{\rho}{2})$. Here δ_0 is the Dirac function at the origin with respect to the measure dV_0 and \tilde{G}_1 is defined by the last equality in (3.14). Set $G_1(\xi, \eta) := \tilde{G}_1(\xi^{-1}\eta)$ for $\xi, \eta \in U$. Then, $G_1(\xi, \eta) \in C^\infty(U \times U)$ and for each $\xi \in U$, $G_1(\xi, \cdot)$ can be naturally extended to a smooth function on M satisfying $G_1(\xi, \eta) = 0$ for $\eta \notin \tilde{U}$. By transformation law (3.11) and left invariance of Y_j , we find that

$$\begin{aligned} L_{g,\mathbb{Q}}(\phi(\xi)^{-1}\phi(\cdot)^{-1}\tilde{G}(\xi, \cdot)) &= \phi(\xi)^{-1}\phi(\cdot)^{-\frac{Q+2}{Q-2}}L_0\tilde{G}(\xi, \cdot) = \phi(\xi)^{-1}\phi(\cdot)^{-\frac{Q+2}{Q-2}}(\delta_0(\xi^{-1}\cdot) + G_1(\xi, \cdot)) \\ &= \delta_\xi + \phi(\xi)^{-1}\phi(\cdot)^{-\frac{Q+2}{Q-2}}G_1(\xi, \cdot), \end{aligned}$$

on U for $\xi \in U$, where δ_ξ is the Dirac function at point ξ with respect to the measure $dV_{g,\mathbb{Q}} = \phi^{\frac{2Q}{Q-2}}dV_0$. Now set

$$(3.15) \quad G(\xi, \eta) := \phi(\xi)^{-1}\phi(\eta)^{-1}\tilde{G}(\xi, \eta) + G_2(\xi, \eta)$$

for $\eta \in M$, where $G_2(\xi, \eta)$ satisfies

$$(3.16) \quad L_{g,\mathbb{Q}}G_2(\xi, \cdot) = -\phi(\xi)^{-1}\phi(\cdot)^{-\frac{Q+2}{Q-2}}G_1(\xi, \cdot).$$

$G_2(\xi, \cdot)$ exists since $L_{g,\mathbb{Q}}$ is invertible in $L^2(M)$. $G_2(\xi, \cdot) \in C^\infty(M)$ for fixed $\xi \in U$ by the subelliptic regularity of $L_{g,\mathbb{Q}}$. $G_2(\cdot, \eta)$ is also in $C^\infty(U)$ by differentiating (3.16) with respect to the variable ξ repeatedly. Now we have $L_{g,\mathbb{Q}}G(\xi, \cdot) = \delta_\xi$, i.e. $G(\xi, \eta)$ is the Green function of $L_{g,\mathbb{Q}}$. By (3.15), $G_{g,\mathbb{Q}}(\xi, \eta) - \rho_{g,\mathbb{Q}}(\xi, \eta) \in C^\infty(U \times U)$. \square

We have the following transformation law of Green functions under the conformal qc transformation.

Proposition 3.7. *Let (M, g, \mathbb{Q}) be a connected, compact, scalar positive spherical qc manifold and $G_{g, \mathbb{Q}}$ be the Green function of the qc Yamabe operator $L_{g, \mathbb{Q}}$. Then*

$$(3.17) \quad G_{\tilde{g}, \mathbb{Q}} = \frac{1}{\phi(\xi)\phi(\eta)} G_{g, \mathbb{Q}}(\xi, \eta)$$

is the Green function of the qc Yamabe operator $L_{\tilde{g}, \mathbb{Q}}$ for $\tilde{g} = \phi^{\frac{4}{\mathbb{Q}-2}} g$.

Proof. By (3.10) and the transformation law (3.11), we find that

$$\begin{aligned} \int_M \frac{G_{g, \mathbb{Q}}(\xi, \eta) L_{\tilde{g}, \mathbb{Q}} u(\eta)}{\phi(\xi)\phi(\eta)} dV_{\tilde{g}, \mathbb{Q}} &= \frac{1}{\phi(\xi)} \int_M \frac{1}{\phi(\eta)} G_{g, \mathbb{Q}}(\xi, \eta) \phi(\eta)^{-\frac{\mathbb{Q}+2}{\mathbb{Q}-2}} L_{g, \mathbb{Q}}(\phi u)(\eta) \phi(\eta)^{\frac{2\mathbb{Q}}{\mathbb{Q}-2}} dV_{g, \mathbb{Q}} \\ &= \frac{1}{\phi(\xi)} \int_M G_{g, \mathbb{Q}}(\xi, \eta) L_{g, \mathbb{Q}}(\phi u)(\eta) dV_{g, \mathbb{Q}} = u(\xi) \end{aligned}$$

for any $u \in C_0^\infty(M)$. The proposition follows from the uniqueness of the Green function. \square

4. AN INVARIANT TENSOR ON A SCALAR POSITIVE SPHERICAL QC MANIFOLD

4.1. A tensor invariant under conformal qc transformations.

Theorem 4.1. *Let (M, g, \mathbb{Q}) be connected, compact, scalar positive, spherical qc manifold, which is not qc equivalent to the standard sphere. Define*

$$\text{can}(g, \mathbb{Q}) := \mathcal{A}_{g, \mathbb{Q}}^2 g,$$

where $\mathcal{A}_{g, \mathbb{Q}}^2(\xi)$ is defined in (1.4) if $g = \phi^{\frac{4}{\mathbb{Q}-2}} g_0$ on a neighborhood U of ξ . Here g_0 is the standard qc metric on \mathcal{H}^n . Then, $\text{can}(g, \mathbb{Q})$ is well-defined and depends only on the conformal class $[g]$ and \mathbb{Q} .

Proof. We will verify that $\mathcal{A}_{g, \mathbb{Q}}$ is independent of the choice of local coordinates and $\mathcal{A}_{g, \mathbb{Q}}^2 g$ is independent of the choice of g in the conformal class $[g]$. Suppose $\tilde{g} = \Phi^{\frac{4}{\mathbb{Q}-2}} g$. Let $U \subset M$ be an open set and let $\rho : U \rightarrow V \subset \mathcal{H}^n$ and $\tilde{\rho} : U \rightarrow \tilde{V} \subset \mathcal{H}^n$ be two coordinate charts such that

$$g = \rho^* \left(\phi_1^{\frac{4}{\mathbb{Q}-2}} g_0 \right), \quad \tilde{g} = \tilde{\rho}^* \left(\phi_2^{\frac{4}{\mathbb{Q}-2}} g_0 \right),$$

for two positive function ϕ_1 and ϕ_2 . Then, $f = \tilde{\rho} \circ \rho^{-1} : V \rightarrow \tilde{V}$ is a qc diffeomorphism of \mathcal{H}^n by the qc Liouville type Theorem 2.2 and

$$f^* g_0|_{\xi'} = \phi^{\frac{4}{\mathbb{Q}-2}}(\xi') g_0|_{\xi'} \quad \text{with } \phi(\xi') = \phi_1(\xi') \phi_2^{-1}(f(\xi')) \Phi(\rho^{-1}(\xi')),$$

for $\xi' \in V$. We claim the following the transformation law of the Green function on the quaternionic Heisenberg group under a conformal qc transformation:

$$(4.1) \quad \frac{1}{\|f(\xi')^{-1} f(\eta')\|^{Q-2}} = \frac{1}{\phi(\xi') \phi(\eta')} \cdot \frac{1}{\|\xi'^{-1} \eta'\|^{Q-2}},$$

for any $\xi', \eta' \in V$. Apply this to $\xi' = \rho(\xi)$, $\eta' = \rho(\eta)$ and $f = \tilde{\rho} \circ \rho^{-1}$ to get

$$\frac{1}{\|\tilde{\rho}(\xi)^{-1} \tilde{\rho}(\eta)\|^{Q-2}} = \frac{1}{\phi(\rho(\xi))} \frac{1}{\phi(\rho(\eta))} \frac{1}{\|\rho(\xi)^{-1} \rho(\eta)\|^{Q-2}}$$

and so

$$\begin{aligned}
\mathcal{A}_{\tilde{g}, \mathbb{Q}}(\xi) &= \lim_{\eta \rightarrow \xi} \left| G_{\tilde{g}, \mathbb{Q}}(\xi, \eta) - \frac{1}{\phi_2(\tilde{\rho}(\xi))\phi_2(\tilde{\rho}(\eta))} \cdot \frac{C_Q}{\|\tilde{\rho}(\xi)^{-1}\tilde{\rho}(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}} \\
&= \lim_{\eta \rightarrow \xi} \left| \frac{G_{g, \mathbb{Q}}(\xi, \eta)}{\Phi(\xi)\Phi(\eta)} - \frac{1}{\Phi(\xi)\Phi(\eta)\phi_1(\rho(\xi))\phi_1(\rho(\eta))} \cdot \frac{C_Q}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}} \\
&= \Phi^{-\frac{2}{Q-2}}(\xi) \lim_{\eta \rightarrow \xi} \left| G_{g, \mathbb{Q}}(\xi, \eta) - \frac{1}{\phi_1(\rho(\xi))\phi_1(\rho(\eta))} \cdot \frac{C_Q}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}} \\
&= \Phi^{-\frac{2}{Q-2}}(\xi) \mathcal{A}_{g, \mathbb{Q}}(\xi).
\end{aligned}$$

Consequently, we have

$$\mathcal{A}_{\tilde{g}, \mathbb{Q}}^2 \tilde{g} = \mathcal{A}_{g, \mathbb{Q}}^2 g.$$

It remains to check (4.1). By qc Liouville type Theorem 2.2, f is a restriction to U of a qc automorphism of \mathcal{H}^n , denoted also by f . By the transformation law (3.11), for functions $\tilde{\phi} := \phi \circ f^{-1}$, $\tilde{u} := u \circ f^{-1}$ on \tilde{V} , we have

$$\begin{aligned}
(4.2) \quad L_0 \left(\tilde{\phi}^{-1} \tilde{u} \right) \Big|_{f(\eta')} &= L_{g, \mathbb{Q}} \left(\phi^{-1} u \right) \Big|_{\eta'} = \phi^{-\frac{Q+2}{Q-2}}(\eta') L_0(u) \Big|_{\eta'}, \\
f^* dV_0 \Big|_{f(\eta')} &= \phi^{\frac{2Q}{Q-2}}(\eta') dV_0 \Big|_{\eta'}.
\end{aligned}$$

The first identity follows from the fact that the qc Yamabe operator is independent of the choice of coordinate charts. Then, by (4.2) and taking transformation $f(\eta') \rightarrow \hat{\eta}$, we find that for any $u \in C_0^\infty(\mathcal{H}^n)$

$$\begin{aligned}
\int_{\mathcal{H}^n} \frac{C_Q \phi(\xi') \phi(\eta')}{\|f(\xi')^{-1} f(\eta')\|^{Q-2}} L_0 u(\eta') dV_0(\eta') &= \int_{\mathcal{H}^n} \frac{C_Q \phi(\xi')}{\|f(\xi')^{-1} f(\eta')\|^{Q-2}} L_0 \left(\tilde{\phi}^{-1} \tilde{u} \right) \Big|_{f(\eta')} f^* dV_0(\eta') \\
&= \int_{\mathcal{H}^n} \frac{C_Q \phi(\xi')}{\|f(\xi')^{-1} \hat{\eta}\|^{Q-2}} L_0 \left(\tilde{\phi}^{-1} \tilde{u} \right) \Big|_{\hat{\eta}} dV_0(\hat{\eta}) = u(\xi').
\end{aligned}$$

Now by the uniqueness of the Green function of L_0 , we find that

$$G_0(\xi', \eta') = \frac{C_Q \phi(\xi') \phi(\eta')}{\|f(\xi')^{-1} f(\eta')\|^{Q-2}}.$$

Thus, (4.1) follows. The theorem is proved. \square

See [24] for the identity (4.1) on the Euclidean space and see [33] on the Heisenberg group.

Corollary 4.1. *($M, \text{can}(g, \mathbb{Q}), \mathbb{Q}$) is a spherical qc manifold if the qc positive mass conjecture is true.*

Proof. Write $g = \phi^{\frac{4}{Q-2}} g_0$ locally. Note that by Proposition 3.4, 3.6 and 3.7, we have

$$\begin{aligned}
G_{g, \mathbb{Q}}(\xi, \eta)^{\frac{4}{Q-2}} g \Big|_{\eta} &= \left\{ \rho_{g, \mathbb{Q}}(\xi, \eta) + A_{g, \mathbb{Q}}(\xi) + O(\|\xi^{-1} \eta\|) \right\}^{\frac{4}{Q-2}} g \Big|_{\eta} \\
&= \frac{\alpha(\xi)}{\|\xi^{-1} \eta\|^4} \cdot \left\{ 1 + \beta(\xi) \|\xi^{-1} \eta\|^{Q-2} + O(\|\xi^{-1} \eta\|^{Q-1}) \right\}^{\frac{4}{Q-2}} g_0 \Big|_{\eta},
\end{aligned}$$

for η near ξ , where

$$\alpha(\xi) = \left(\frac{C_Q}{\phi(\xi)} \right)^{\frac{4}{Q-2}}, \quad \beta(\xi) = \frac{A_{g, \mathbb{Q}}(\xi) \phi(\xi)^2}{C_Q}.$$

Now choose the conformal qc mapping $C_\xi := R \circ D_r \circ \tau_{\xi^{-1}}$ on a neighborhood of ξ to \mathcal{H}^n as required in the qc positive mass conjecture, where $r = \alpha(\xi)^{-\frac{1}{2}}$. It is easy to see that $C_\xi(\xi) = \infty$ and

$$\begin{aligned} \left(C_\xi^{-1} \right)^* \left(G_{g, \mathbb{Q}}(\xi, \eta)^{\frac{4}{Q-2}} g \right) \Big|_{\tilde{\eta}} &= R^* \circ D_{r^{-1}}^* \circ \tau_\xi^* \left(G_{g, \mathbb{Q}}(\xi, \eta)^{\frac{4}{Q-2}} g \right) \Big|_{\tilde{\eta}} \\ &= (1 + \beta(\xi) r^{-Q+2} \|\tilde{\eta}\|^{-Q+2} + O(\|\tilde{\eta}\|^{-Q+1}))^{\frac{4}{Q-2}} g_0|_{\tilde{\eta}}, \end{aligned}$$

by Corollary 2.2, where $\tilde{\eta} = C_\xi(\eta)$. The qc positive mass conjecture promises $\beta(\xi) r^{-Q+2}$ to be positive. So, $\mathcal{A}_{g, \mathbb{Q}}$ is non-vanishing. \square

Remark 4.1. Let (M, g, \mathbb{Q}) and $(\tilde{M}, \tilde{g}, \tilde{\mathbb{Q}})$ be two connected, compact, scalar positive spherical qc manifolds with $\dim M = \dim \tilde{M}$, which are both not qc equivalent to the standard sphere $(S^{4n+3}, g_S, \mathbb{Q}_S)$. Suppose $f : (M, g, \mathbb{Q}) \rightarrow (\tilde{M}, \tilde{g}, \tilde{\mathbb{Q}})$ be a locally qc diffeomorphism. Then,

$$f^* \text{can}(\tilde{g}, \tilde{\mathbb{Q}})(X, Y) \geq \text{can}(g, \mathbb{Q})(X, Y),$$

for any tangent vector $X, Y \in TM$. This can be shown as in the CR case (cf. Proposition 3.4 in [33]).

4.2. Scalar positivity of the connected sum of two scalar positive spherical qc manifolds. Scalar positive spherical qc manifolds are abundant by the following proposition.

Proposition 4.1. *If t is sufficiently small, the connected sum $(M_{t, \sigma, A}, g, \mathbb{Q}_{t, \sigma, A})$ is scalar positive.*

Proposition 4.1 follows from the following proposition.

Proposition 4.2. *If t is sufficiently small, we have $\lambda(M_{t, \sigma, A}, g, \mathbb{Q}_{t, \sigma, A}) > 0$.*

Proof. See [23] for the Riemannian case. We provide more details that are different from the Riemannian case, compared to the proof of the CR case in [33]. Let

$$M_0 = M_1 \setminus \{\xi_1\} \cup M_2 \setminus \{\xi_2\},$$

and let \hat{g} be a spherical qc metric on M_0 . Then, by multiplying a positive function $\mu \in C^\infty(M) \setminus \{\xi_1, \xi_2\}$, we can assume $g = \mu \hat{g}$ satisfying

$$(\psi_i^{-1})^* g|_\xi = \frac{g_0|_\xi}{\|\xi\|^2} \quad \text{on } B_{\mathcal{H}}(0, 2) \setminus \{0\},$$

where $\psi_i : U_i \rightarrow B_{\mathcal{H}}(0, 2)$, $i = 1, 2$, are coordinate charts in (2.44). It is easy to see that gluing mapping $\Psi_{t, \sigma, A}$ in (2.45) preserves the metric $\frac{g_0|_\xi}{\|\xi\|^2}$ on $t < \|\xi\| < \frac{1}{t}$, $0 < \frac{2}{3}t < 1$, by the transformation law and Corollary 2.2. $\frac{g_0|_\xi}{\|\xi\|^2}$ is invariant under the rotation A , transformation σ and the inversion R . Hence we can glue g by $\Psi_{t, \sigma, A}$ to obtain a spherical qc metric that coincides with g on $M_1 \setminus \overline{U_1(t)} \cup M_2 \setminus \overline{U_2(t)}$. We denote the resulting qc metric also by g by abuse of notations. We denote the connected sum by (M_t, g, \mathbb{Q}_t) . Here we omit the subscripts σ and A for simplicity. The scalar curvature of g on $B_{\mathcal{H}}(0, 1) \setminus \{0\}$ is

$$s_{g, \mathbb{Q}}(\xi) = \frac{(Q-2)(Q+2)}{2} \cdot \frac{|y|^2}{\|\xi\|^2}, \quad \text{for } \xi = (y, t) \in \mathcal{H}^n,$$

by (A.8) in Appendix.

(M_0, g, \mathbb{Q}) has two cylindrical ends. We can identify the ball with cylindrical end by the mapping

$$(4.3) \quad \begin{aligned} \phi : B(0, 1) &\longrightarrow [0, \infty) \times \Sigma^n \\ \xi = D_{e^{-u}}(\eta) &\longmapsto \left(\ln \frac{1}{\|\xi\|}, \frac{\xi}{\|\xi\|} \right) = (u, \eta), \end{aligned}$$

where $\Sigma^n = \{\eta \in \mathcal{H}^n; \|\eta\| = 1\}$ is diffeomorphic to the sphere S^{4n+2} . Define a Carnot-Carathéodory metric

$$\tilde{g}|_\xi = (\phi^{-1})^* \left(\frac{g_0|_\xi}{\|\xi\|^2} \right)$$

on $[0, \infty) \times \Sigma^n$ and $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ is a compatible contact form. $(B_{\mathcal{H}}(0, 1) \setminus \{0\}, \frac{g_0}{\|\xi\|^2}, \mathbb{Q})$ is qc equivalent to $([0, \infty) \times \Sigma^n, \tilde{g}, \mathbb{Q})$. Since

$$(\psi_i^{-1})^* dV_{g, \mathbb{Q}} = \frac{dV_0}{\|\xi\|^Q}$$

is invariant under rescaling, it is easy to see that the measure $\tilde{\theta}_1 \wedge \tilde{\theta}_2 \wedge \tilde{\theta}_3 \wedge (d\tilde{\theta}_s)^{2n}$ is invariant under translation $(u', \xi) \rightarrow (u' + u_0, \xi)$ on $[0, \infty) \times \Sigma^n$. As a measure, we have

$$(4.4) \quad \tilde{\theta}_1 \wedge \tilde{\theta}_2 \wedge \tilde{\theta}_3 \wedge (d\tilde{\theta}_s)^{2n} = dudS_{\Sigma^n},$$

where dS_{Σ^n} is a measure on Σ^n . set

$$l = \ln \frac{1}{t},$$

and write

$$(M_0, g, \mathbb{Q}) = ([0, \infty) \times \Sigma^n, \tilde{g}, \mathbb{Q}) \cup (\hat{M}, g, \mathbb{Q}) \cup ([0, \infty) \times \Sigma^n, \tilde{g}, \mathbb{Q}),$$

where $\hat{M} = M \setminus (U_1(1) \cup U_2(1))$. We identify two pieces of $(0, l) \times \Sigma^n, \tilde{g}, \mathbb{Q})$ to get (M_t, g, \mathbb{Q}_t) .

Denote by y_η the y -coordinate of $\eta \in \Sigma^n$, i.e. we can write $\eta = (y_\eta, t_\eta) \in \mathcal{H}^n$ for some $t_\eta \in \mathbb{R}^3$. Then,

$$(4.5) \quad |\nabla_{\tilde{g}} u| = \left| \nabla_g \left(\ln \frac{1}{\|\xi\|} \right) \right| = \frac{1}{2} |y_\xi|^2 = \frac{1}{2} e^{-2u} |y_\eta|^2,$$

where $\xi = (y_\xi, t_\xi)$ for some t_ξ , and

$$(4.6) \quad s_{\tilde{g}, \mathbb{Q}}(u, \eta) = \frac{(Q-2)(Q+2)}{2} |y_\eta|^2,$$

by (A.7) and (A.8) in the appendix. By the definition of the Yamabe invariant $\lambda(M_t, g, \mathbb{Q}_t)$, we can find a positive function $f_l \in C^\infty(M_t)$ such that

$$(4.7) \quad \int_{M_t} (b_n |\nabla_g f_l|^2 + s_{g, \mathbb{Q}_t} f_l^2) dV_{g, \mathbb{Q}_t} < \lambda(M_t, g, \mathbb{Q}_t) + \frac{1}{l},$$

and

$$(4.8) \quad \int_{M_t} f_l^{\frac{2Q}{Q-2}} dV_{g, \mathbb{Q}_t} = 1.$$

Put $A_1 = -\min \{0, \min_{x \in \hat{M}} s_{\hat{g}, \mathbb{Q}}\} \text{Vol}(\hat{M})^{\frac{4}{Q-2}}$, which is uniformly bounded by $\text{Vol}(M, g)$. Thus by using Hölder's inequality we get from (4.7) that

$$\int_{[0, l] \times \Sigma^n} (b_n |\nabla_{\tilde{g}} f_l|^2 + s_{\tilde{g}, \mathbb{Q}_t} f_l^2) dudS_{\Sigma^n} < \lambda(M_t, g, \mathbb{Q}_t) + \frac{1}{l} + A_1,$$

(cf. Lemma 6.2 in [23]). Note that $s_{\tilde{g}, \mathbb{Q}_t}$ is nonnegative on $[0, \infty) \times \Sigma^n$ by (4.6). Therefore, there exists $l_* \in [0, l]$ such that

$$\int_{l_* \times \Sigma^n} (b_n |\nabla_{\tilde{g}} f_l|^2 + s_{\tilde{g}, \mathbb{Q}_t} f_l^2) dS_{\Sigma^n} < \frac{\lambda(M_t, g, \mathbb{Q}_t) + \frac{1}{l} + A_1}{l},$$

i.e. we have the estimate

$$(4.9) \quad \int_{\Sigma^n} (|\nabla_{\tilde{g}} f_l(l_*, \eta)|^2 + |y_\eta|^2 f_l^2(l_*, \eta)) dS_{\Sigma^n}(\eta) < \frac{C}{l},$$

by the scalar curvature of g in (4.6), where C is a constant independent of l (because the qc Yamabe invariants $\lambda(M_t, g, \mathbb{Q}_t)$ for $t > 1$ have a uniform upper bound by choosing a test function). It is different from the Riemannian case that the scalar curvature $\frac{g_0|_\xi}{\|\xi\|^2}$ is not constant. But it is still independent of the variable u . Now define a Lipschitz function F_l on M_0 by $F_l = f_l$ on $[0, l_*) \times \Sigma^n \cup \hat{M} \cup [0, l - l_*) \times \Sigma^n$ and

$$(4.10) \quad F_l(u, x) = \begin{cases} (l_* + 1 - u) \tilde{f}_l(u, x) & \text{for } (u, x) \in [l_*, l_* + 1] \times \Sigma^n, \\ 0 & \text{for } (u, x) \in [l_* + 1, \infty) \times \Sigma^n, \end{cases}$$

where $\tilde{f}_l(u, x) = f_l(l_*, x)$ and similarly on $[l - l_*, \infty) \times \Sigma^n$.

By definition, $|\nabla_{\tilde{g}} F_l| = |\nabla_{\tilde{g}} f_l|$ and $F_l^2 = f_l^2$ hold on $[0, l_*) \times \Sigma^n \cup \hat{M} \cup [0, l - l_*) \times \Sigma^n$. On the other hand, note that $|\nabla_{\tilde{g}} F_l| \leq |\nabla_{\tilde{g}} u| |\tilde{f}_l| + |\nabla_{\tilde{g}} \tilde{f}_l|$ pointwisely on $(l_*, l_* + 1) \times \Sigma^n$ by definition. By (4.5), (4.6) and estimate (4.9), we find that

$$\begin{aligned} \int_{(l_*, l_* + 1) \times \Sigma^n} (b_n |\nabla_{\tilde{g}} F_l|^2 + s_{\tilde{g}, \mathbb{Q}} F_l^2) dudS_{\Sigma^n} &\leq C' \int_{(l_*, l_* + 1) \times \Sigma^n} (|\nabla_{\tilde{g}} \tilde{f}_l|^2 + |y_\eta|^2 \tilde{f}_l^2) dudS_{\Sigma^n}(\eta) \\ &\leq C'' \int_{\Sigma^n} (|\nabla_{\tilde{g}} f_l(l_*, \eta)|^2 + |y_\eta|^2 f_l^2(l_*, \eta)) dS_{\Sigma^n}(\eta) \leq \frac{B'}{l} \end{aligned}$$

by $|\nabla_{\tilde{g}} \tilde{f}_l(u, \eta)| = |\nabla_{\tilde{g}} \tilde{f}_l(l_*, \eta)| \leq |\nabla_{\tilde{g}} f_l(l_*, \eta)|$ pointwisely, since horizontal subspace H and \tilde{g} are invariant under the translation $(u, \eta) \rightarrow (u + u_0, \eta)$ and \tilde{f}_l is independent of u . Therefore, we get

$$\int_{M_0} (b_n |\nabla_g F_l|^2 + s_{g, \mathbb{Q}} F_l^2) dV_{g, \mathbb{Q}} < \lambda(M_t, g, \mathbb{Q}_t) + \frac{B}{l},$$

for some constant B independent of l .

Obviously from (4.8) and the definition of F_l , we get

$$\int_{M_0} F_l^{\frac{2Q}{Q-2}} dV_{g, \mathbb{Q}} > 1.$$

Therefore,

$$(4.11) \quad \inf_{F > 0} \frac{\int_{M_0} (b_n |\nabla_g F|^2 + s_{g, \mathbb{Q}} F^2) dV_{g, \mathbb{Q}}}{\left(\int_{M_0} F^{\frac{2Q}{Q-2}} dV_{g, \mathbb{Q}} \right)^{\frac{Q-2}{Q}}} < \lambda(M_t, g, \mathbb{Q}_t) + \frac{B}{l},$$

where the infimum is taken over all nonnegative Lipschitz functions with compact support. It follows from the definition of the Yamabe invariant that the left side is greater than or equal to $\lambda(M, g, \mathbb{Q})$. If (M, g, \mathbb{Q}) is a disjoint union of (M_1, g_1, \mathbb{Q}) and (M_2, g_2, \mathbb{Q}) , we have

$$(4.12) \quad \lambda(M, g, \mathbb{Q}) = \min\{\lambda(M_1, g_1, \mathbb{Q}), \lambda(M_2, g_2, \mathbb{Q})\},$$

by the definition of the qc Yamabe invariant. From (4.11) and (4.12), $\lambda(M_t, g, \mathbb{Q}_t)$ is positive if l is sufficiently large, i.e. t is sufficiently small. We complete the proof. \square

5. THE CONVEX COCOMPACT DISCRETE SUBGROUPS OF $\mathrm{Sp}(n+1, 1)$

5.1. Convex cocompact subgroups of $\mathrm{Sp}(n+1, 1)$. A group G is called *discrete* if the topology on G is the discrete topology. We say that G acts *discontinuously* on a space X at point q if there is a neighborhood U of q , such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$.

Let Γ be a discrete subgroup of $\mathrm{Sp}(n+1, 1)$. Choose $q \in B^{4n+4}$. We define the *limit set* of Γ by

$$\Lambda(\Gamma) := \overline{\Gamma q} \cap S^{4n+3},$$

where $\overline{\Gamma q}$ is the closure of the orbit of q under Γ . It is known that $\Lambda(\Gamma)$ does not depend on the choice of $q \in B^{4n+4}$ (cf. Proposition 1.4 and Proposition 2.9 in [9]). The limit set $\Lambda(\Gamma)$ of all limit points is closed and invariant under Γ . The *radial limit set* of Γ is

$$\Lambda^r(\Gamma) := \left\{ \xi \in \Lambda(\Gamma) \mid \liminf_{T \rightarrow \infty} d(\xi_T, \gamma(0)) < \infty, \gamma \in \Gamma \right\},$$

where ξ_T refers to the point on the ray from 0 to ξ for which $d(0, \xi_T) = T$ and $d(\cdot, \cdot)$ is the quaternionic hyperbolic distance.

$$\Omega(\Gamma) := S^{4n+3} \setminus \Lambda(\Gamma)$$

is the maximal domain in S^{4n+3} on which Γ acts properly discontinuously by Proposition 8.5 in [9]. Γ is called a *Kleinian group* if $\Omega(\Gamma)$ is non empty. A Kleinian group is called *elementary* if $\Lambda(\Gamma)$ contains at most two points. The Kleinian manifold associated to Γ is defined to be

$$\bar{M}_\Gamma = (B^{4n+4} \cup \Omega(\Gamma)) / \Gamma.$$

It is known that $\Omega(\Gamma)/\Gamma$ is a smooth manifold and \bar{M}_Γ is a manifold with boundary (cf. the proof of Corollary 11.11 in [9]). Γ is called *convex cocompact* if \bar{M}_Γ is a compact manifold with boundary. In this case, $\Omega(\Gamma)/\Gamma$ is a compact smooth manifold.

Two basic properties of convex cocompact groups are given in the following proposition.

Proposition 5.1. (cf. p. 528 in [6]) *Suppose that Γ is a convex cocompact group of $\mathrm{Sp}(n+1, 1)$. Then,*

- (i) *The radial limit set coincides with the limit set.*
- (ii) *Any small deformation of the inclusion $\iota : \Gamma \rightarrow \mathrm{Sp}(n+1, 1)$ maps Γ isomorphically to a convex cocompact group.*

An interesting class of convex cocompact groups can be obtained as follows. For a convex cocompact group Γ in $\mathrm{Sp}(n+1, 1)$, there is a large family of infinitesimal deformations of Γ and these are all unobstructed. The small deformations give convex cocompact groups by (ii) above.

Let $\{C_i, C'_i\}_{i=1}^k$ be the boundary of mutually disjoint balls $\{D_i, D'_i\}_{i=1}^k$, where $D_i = B_{\mathcal{H}}(\xi_i, r_i)$ and $D'_i = B_{\mathcal{H}}(\xi'_i, r'_i)$, for some points $\xi_i, \xi'_i \in \mathcal{H}^n$, $r_i, r'_i > 0$, $i = 1, \dots, k$. There always exist group elements $\gamma_i \in \mathrm{Sp}(n+1, 1)$ such that

$$\gamma_i(\mathcal{H}^n \setminus \bar{D}_i) = D'_i$$

(the “dilations” with the origin and infinity played by the centers of balls D_i and D'_i , respectively). Then $\{\gamma_i\}$ generates a convex cocompact subgroup Γ which is isomorphic to the free

group of rank k . As in the Riemannian case, we call such group the *Schottky group*. It is easy to see that $S^{4n+3} \setminus \cup_1^k (\overline{D_i} \cup D'_i)$ is the fundamental domain for Γ . $\Omega(\Gamma)/\Gamma$ is diffeomorphic to

$$(5.1) \quad k = (S^1 \times S^{4n+2}) \# \dots \# (S^1 \times S^{4n+2}),$$

$\#$ is the connected sum operation defined by (2.46).

See also [2] for other interesting examples of convex cocompact groups.

5.2. The Patterson-Sullivan measure.

Theorem 5.1. (cf. p. 532 in [6]) *For any convex cocompact Kleinian group $\Gamma \subset \mathrm{Sp}(n+1, 1)$, there exists a probability measure μ_Γ supported on $\Lambda(\Gamma)$ such that*

$$(5.2) \quad \gamma^* \mu_\Gamma = |\gamma'|^{\delta(\Gamma)} \mu_\Gamma$$

for any $\gamma \in \Gamma$, where $|\gamma'|$ is the conformal factor.

See also [10] and [33] for Patterson-Sullivan measure for the complex case. We need to know the explicit conformal factor $|\gamma'|$ for our purpose later. Fix a point $q \in \mathbb{H}^{n+1}$, the series $\sum_{\gamma \in \Gamma} e^{-\frac{s}{2} \cdot d(p, \gamma(q))}$ converges for $s > \delta(\Gamma)$ for any $p \in B^{4n+4}$, and diverges for any $s < \delta(\Gamma)$. Fix a reference point $0 \in B^{4n+4}$. Let us recall the definition of Patterson-Sullivan measure in [30]. Define a family of measures as

$$(5.3) \quad \mu_{s,p} := \frac{\sum_{\gamma \in \Gamma} e^{-\frac{1}{2}s \cdot d(p, \gamma(q))} \delta_{\gamma(q)}}{\sum_{\gamma \in \Gamma} e^{-\frac{1}{2}s \cdot d(0, \gamma(q))}},$$

where $\delta_{\gamma(q)}$ is the Dirac measure supported at point $\gamma(q)$. For each $s > \delta(\Gamma)$, this is a finite positive measure concentrated on $\Gamma q \subset \overline{\Gamma q}$. The set of all probability measures on $\overline{\Gamma q}$ is compact (cf. p. 532 in [6]), and so there is a sequence s_i approaching $\delta(\Gamma)$ from above such that $\mu_{s_i,p}$ approaches a limit $\mu_{s,p}$. After rewriting the coefficients, we may assume that the denominator in (5.3) diverges at $s = \delta(\Gamma)$. Thus, we replace the above expression by

$$(5.4) \quad \mu_{s,p} = \frac{\sum_{\gamma \in \Gamma} a_\gamma e^{-\frac{1}{2}s \cdot d(p, \gamma(q))} \delta_{\gamma(q)}}{\sum_{\gamma \in \Gamma} a_\gamma e^{-\frac{1}{2}s \cdot d(0, \gamma(q))}}$$

with a_γ 's so chosen that the denominator converges for $s > \delta(\Gamma)$ and diverges for $s \leq \delta(\Gamma)$. The denominator of this expression will be denoted by $L(s, 0)$. The definition of the measure μ_s does not depend on $p \in B^{4n+4}$ and the choice of a_γ (cf. p. 532 [6]). The *Patterson-Sullivan measure* is the weak limit of these measures:

$$\mu_{\Gamma,p} = \lim_{s_i \rightarrow \delta(\Gamma)^+} \mu_{s_i,p}.$$

Since we use the right action of matrix, here $\gamma^{-1}(\gamma'(q)) = (\gamma' \gamma^{-1})(q)$. For any $\gamma \in \mathrm{Sp}(n+1, 1)$ and any $f \in C(\overline{B})$, we have

$$\begin{aligned} (\gamma^* \mu_{s_i,p})(f) &= \frac{\sum_{\gamma' \in \Gamma} a_{\gamma'} e^{-\frac{1}{2}s_i \cdot d(p, \gamma'(q))} \gamma^* \delta_{\gamma'(q)}(f)}{L(s_i, 0)} \\ &= \frac{\sum_{\gamma' \in \Gamma} a_{\gamma'} e^{-\frac{1}{2}s_i \cdot d(\gamma^{-1}(p), \gamma^{-1}(\gamma'(q)))} f(\gamma^{-1}(\gamma'(q)))}{L(s_i, 0)} \\ &= \frac{\sum_{\gamma' \in \Gamma} a_{\gamma' \gamma} e^{-\frac{1}{2}s_i \cdot d(\gamma^{-1}(p), \gamma'(q))} f(\gamma'(q))}{L(s_i, 0)} = \mu_{s_i, \gamma^{-1}(p)}(f) \end{aligned}$$

by the invariance of the quaternionic hyperbolic distance $d(\cdot, \cdot)$ under the action of $\mathrm{Sp}(n+1, 1)$. It is easy to see that $\{a_{\gamma'\gamma}\}$ is also such sequence satisfying the definition for fixed γ . Letting $s_i \rightarrow \delta^+$, we get

$$\gamma^* \mu_{\Gamma, p} = \mu_{\Gamma, \gamma^{-1}(p)}.$$

Recall that we have the following the Radon-Nikodym relation (cf. p.77, p.81 in [38]):

$$\left. \frac{d\mu_{\Gamma, \gamma^{-1}(p)}}{d\mu_{\Gamma, p}} \right|_{\xi} = e^{-\frac{\delta}{2} \rho_{p, \xi}(\gamma^{-1}(p))}, \quad \xi \in S^{4n+3},$$

where $\rho_{p, \xi}(q)$ is the Buseman function at $\xi \in \Lambda(\Gamma)$ normalized such that $\rho_{p, \xi}(p) = 0$. It follows from the formula of Buseman function (cf. p. 81 in [38]),

$$\rho_{p, \xi}(q) = \ln \frac{|1 - \langle p, p \rangle| |1 - \langle q, \xi \rangle|^2}{|1 - \langle q, q \rangle| |1 - \langle p, \xi \rangle|^2},$$

that

$$\left. \frac{d\mu_{\Gamma, \gamma^{-1}(p)}}{d\mu_{\Gamma, p}} \right|_{\xi} = \left(\frac{|1 - \langle \gamma^{-1}(p), \gamma^{-1}(p) \rangle|^{\frac{1}{2}} |1 - \langle p, \xi \rangle|}{|1 - \langle p, p \rangle|^{\frac{1}{2}} |1 - \langle \gamma^{-1}(p), \xi \rangle|} \right)^{\delta(\Gamma)},$$

for $\xi \in S^{4n+3}$. Then, for $p = 0$, we have

$$\begin{aligned} \left. \frac{d\mu_{\Gamma, \gamma^{-1}(0)}}{d\mu_{\Gamma, 0}} \right|_{\xi} &= \lim_{\eta \rightarrow \xi} \left(\frac{|1 - \langle \gamma^{-1}(0), \gamma^{-1}(0) \rangle|^{\frac{1}{2}} |1 - \langle 0, \eta \rangle|}{|1 - \langle 0, 0 \rangle|^{\frac{1}{2}} |1 - \langle \gamma^{-1}(0), \eta \rangle|} \right)^{\delta(\Gamma)} \\ &= \lim_{\eta \rightarrow \xi} \left| \frac{(0, \eta)}{(\gamma^{-1}(0), \eta)} \right|^{\delta(\Gamma)} = \lim_{\eta \rightarrow \xi} \left| \frac{(0, \eta)}{(0, \gamma(\eta))} \right|^{\delta(\Gamma)} \\ &= \lim_{\eta \rightarrow \xi} \left| \frac{1 - \langle \gamma(\eta), \gamma(\eta) \rangle}{1 - \langle \eta, \eta \rangle} \right|^{\frac{\delta(\Gamma)}{2}} = \frac{1}{|[(\xi, 1)\gamma]_{n+2}|^{\delta(\Gamma)}}, \end{aligned}$$

by (2.37), where $\eta \in B^{4n+4}$ and $|\langle \cdot, \cdot \rangle|$ defined in (2.22)-(2.23) is invariant under $\mathrm{Sp}(n+1, 1)$. So if we define $\mu_{\Gamma} := \mu_{\Gamma, 0}$, we get

$$\gamma^* d\mu_{\Gamma}(\xi) = \frac{1}{|[(\xi, 1)\gamma]_{n+2}|^{\delta(\Gamma)}} d\mu_{\Gamma}(\xi).$$

Then, we have

Proposition 5.2. *For any $\gamma \in \Gamma$, the conformal factor*

$$|\gamma'(\xi)| = \frac{1}{|[(\xi, 1)\gamma]_{n+2}|}, \quad \text{for } \xi \in S^{4n+3}.$$

Theorem 5.2. (cf. p. 533 in [6]) If $\Gamma \in \mathrm{Sp}(n+1, 1)$ is a convex cocompact group which is not contained in any proper parabolic subgroup. Then the measure μ_{Γ} coincides with $\delta(\Gamma)$ -dimensional Hausdorff measure concentrated on $\Lambda(\Gamma)$, i.e. there exist constants $C_1 < C_2$ and R such that if $\xi \in \Lambda(\Gamma)$ and $r < R$, then

$$C_1 \leq \frac{\mu_{\Gamma}(B_r(\xi))}{r^{\delta(\Gamma)}} \leq C_2,$$

where $B_r(\xi)$ is the ball in S^{4n+3} with radius r under the Carnot-Carathéodory distance d_{cc} . Here $d_{cc}(p, q) = \inf_{\gamma} \int_0^1 |\gamma'(t)| dt$ for any $p, q \in S^{4n+3}$, where $\gamma : [0, 1] \rightarrow S^{4n+3}$ are Lipschitzian horizontal curves, i.e. $\gamma'(t) \in H_{\gamma(t)}$ almost everywhere.

6. AN INVARIANT QC METRIC OF NAYATANI TYPE

When the spherical qc manifold is $\Omega(\Gamma)/\Gamma$ for some convex cocompact subgroup Γ of $\mathrm{Sp}(n+1, 1)$, we can construct an invariant qc metric g_Γ , which is the qc generalization of Nayatani's canonical metric in conformal geometry [26].

Define a C^∞ function on $\Omega(\Gamma)$ by

$$\phi_\Gamma(\xi) := \left(\int_{\Lambda(\Gamma)} G_S^\kappa(\xi, \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{1}{\kappa}}, \quad \kappa = \frac{2\delta(\Gamma)}{Q-2}.$$

Set $g_\Gamma := \phi_\Gamma^{\frac{4}{Q-2}} g_S$. Since

$$G_S(\gamma(\xi), \gamma(\zeta)) = |\gamma'(\xi)|^{-\frac{Q-2}{2}} |\gamma'(\zeta)|^{-\frac{Q-2}{2}} G_S(\xi, \zeta)$$

by the notation $|\gamma'|$ in Proposition 5.2, Proposition 2.3 and the transformation law of Green functions (3.17), we have

$$\begin{aligned} (6.1) \quad \phi_\Gamma(\gamma(\xi)) &= \left(\int_{\Lambda(\Gamma)} G_S^{\frac{2\delta(\Gamma)}{Q-2}}(\gamma(\xi), \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{Q-2}{2\delta(\Gamma)}} = \left(\int_{\Lambda(\Gamma)} G_S^{\frac{2\delta(\Gamma)}{Q-2}}(\gamma(\xi), \gamma(\zeta)) d\gamma^* \mu_\Gamma(\zeta) \right)^{\frac{Q-2}{2\delta(\Gamma)}} \\ &= \left(\int_{\Lambda(\Gamma)} |\gamma'(\xi)|^{-\delta(\Gamma)} G_S^{\frac{2\delta(\Gamma)}{Q-2}}(\xi, \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{Q-2}{2\delta(\Gamma)}} = |\gamma'(\xi)|^{-\frac{Q-2}{2}} \phi_\Gamma(\xi) \end{aligned}$$

by using formula (5.2). Therefore, (6.1) together with Proposition 2.3 and Proposition 5.2 implies that $\gamma^* g_\Gamma = g_\Gamma$. So it induces a spherical qc metric on $\Omega(\Gamma)/\Gamma$.

Theorem 6.1. *Let Γ be a convex cocompact subgroup of $\mathrm{Sp}(n+1, 1)$ such that $\Lambda(\Gamma) \neq \{\text{point}\}$. Then, if $\delta(\Gamma) < 2n+2$ (resp. $\delta(\Gamma) > 2n+2$, resp. $\delta(\Gamma) = 2n+2$), the scalar curvature of $(\Omega(\Gamma)/\Gamma, g_\Gamma, \mathbb{Q})$ is positive (resp. negative, resp. zero) everywhere.*

Proof. To calculate its scalar curvature, pull back the qc metric g_Γ locally to the quaternionic Heisenberg group by the quaternionic Cayley transformation. Without loss of generality, we can assume that the southern point $(0, \dots, 0, -1)$ of the sphere S^{4n+3} is contained in $\Omega(\Gamma)$. Let $\xi \in \Omega(\Gamma)$. Under the stereographic projection F defined by (2.28), we have

$$(6.2) \quad G_S(F^{-1}(\xi), F^{-1}(\xi')) = \left(\frac{1}{2|1 + p_{n+1}|^2} \right)^{-\frac{Q-2}{4}} \left(\frac{1}{2|1 + p'_{n+1}|^2} \right)^{-\frac{Q-2}{4}} G_0(\xi, \xi'),$$

by Corollary 2.1 and Proposition 3.6, where $(p, p_{n+1}) = F^{-1}(\xi)$, $(p', p'_{n+1}) = F^{-1}(\xi') \in S^{4n+3}$.

Define the set $\tilde{\Lambda}(\Gamma)$ and the measure $\tilde{\mu}_\Gamma$ on \mathcal{H}^n by

$$(6.3) \quad \tilde{\Lambda}(\Gamma) := F(\Lambda(\Gamma)), \quad \tilde{\mu}_\Gamma(\xi') := \left(\frac{1}{2|1 + p'_{n+1}|^2} \right)^{-\frac{\delta(\Gamma)}{2}} (F^{-1})^* \mu_\Gamma(\xi'),$$

and the metric \tilde{g}_Γ on $\mathcal{H}^n \setminus \tilde{\Lambda}(\Gamma)$ by

$$\tilde{g}_\Gamma = (F^{-1})^* g_\Gamma.$$

By the stereographic projection F , Γ induces an action on \mathcal{H}^n . It follows that the set $\tilde{\Lambda}(\Gamma)$ and the metric \tilde{g}_Γ on \mathcal{H}^n are both invariant under the action of Γ .

Now we begin to calculate its scalar curvature. Write $\tilde{g}_\Gamma|_\xi = u^{\frac{4}{Q-2}}(\xi)g_0|_\xi$, with

$$u(\xi) = \left(\frac{1}{2|1+p_{n+1}|^2} \right)^{\frac{Q-2}{4}} \left(\int_{\Lambda(\Gamma)} G_S^\kappa(F^{-1}(\xi), \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{1}{\kappa}} = \left(\int_{\tilde{\Lambda}(\Gamma)} G_0^\kappa(\xi, \xi') d\tilde{\mu}_\Gamma(\xi') \right)^{\frac{1}{\kappa}},$$

by using (6.2), Corollary 2.1 and the definition of the measure $\tilde{\mu}_\Gamma$ in (6.3). Denote

$$\tilde{g}_\Gamma|_\xi := e^{2f(\xi)}g_0|_\xi, \quad \phi(\xi, \eta) := G_0^{\frac{2}{2-Q}} = C_Q^{\frac{2}{2-Q}} \|\xi^{-1}\eta\|^2$$

for $\xi, \eta \in \mathcal{H}^n$. Then

$$f(\xi) = \frac{1}{\delta(\Gamma)} \ln \left(\int_{\tilde{\Lambda}(\Gamma)} \phi(\xi, \eta)^{-\delta(\Gamma)} d\tilde{\mu}_\Gamma(\eta) \right),$$

and the scalar curvature of \tilde{g}_Γ is

$$s_{\tilde{g}_\Gamma, \mathbb{Q}} = 2(Q+2)e^{-2f(\xi)} \left(\Delta_0 f - \sum_{j=1}^{4n} \frac{Q-2}{4} (Y_j f)^2 \right)$$

by Corollary 3.1. Note that,

$$Y_j f(\xi) = - \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1}(\xi) Y_j \phi_\eta(\xi) d\nu(\eta),$$

where $\phi_\eta(\cdot) := \phi(\cdot, \eta)$ and the measure

$$(6.4) \quad d\nu(\eta) := \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta(\xi)^{-\delta} d\tilde{\mu}_\Gamma(\xi) \right)^{-1} \phi_\eta^{-\delta}(\xi) d\tilde{\mu}_\Gamma(\xi)$$

depends on ξ , where $\delta := \delta(\Gamma)$. Then,

$$Y_j Y_j f(\xi) = - \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j Y_j \phi_\eta d\nu + (\delta+1) \int_{\tilde{\Lambda}(\Gamma)} (\phi_\eta^{-1} Y_j \phi_\eta)^2 d\nu - \delta \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \right)^2,$$

and so

(6.5)

$$\begin{aligned} \Delta_0 f - \frac{Q-2}{4} \sum_{j=1}^{4n} (Y_j f)^2 &= - \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} \Delta_0 \phi_\eta d\nu - \frac{(\delta+1)}{2} \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-2} \sum_{j=1}^{4n} |Y_j \phi_\eta|^2 d\nu \\ &\quad + \frac{\delta}{2} \sum_{j=1}^{4n} \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \right)^2 - \frac{Q-2}{4} \sum_{j=1}^{4n} \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \right)^2 \\ &= \left(\frac{Q}{4} - \frac{\delta+1}{2} \right) \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-2} \sum_{j=1}^{4n} |Y_j \phi_\eta|^2 d\nu - \sum_{j=1}^{4n} \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \right)^2 \right) \\ &\quad - \frac{Q}{4} \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-2} \sum_{j=1}^{4n} |Y_j \phi_\eta|^2 d\nu - \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} \Delta_0 \phi_\eta d\nu, \end{aligned}$$

by $\Delta_0 = -\frac{1}{2} \sum_{j=1}^{4n} Y_j Y_j$. Direct calculation gives

$$\Delta_0 \phi_\eta = \frac{2}{2-Q} G_0^{\frac{Q}{2-Q}} \Delta_0 G_0 - \frac{Q}{(2-Q)^2} G_0^{\frac{2Q-2}{2-Q}} \sum_{j=1}^{4n} |Y_j G_0|^2,$$

and

$$\sum_{j=1}^{4n} |Y_j \phi_\eta|^2 = \frac{4}{(2-Q)^2} G_0^{\frac{2Q}{2-Q}} \sum_{j=1}^{4n} |Y_j G_0|^2.$$

Moreover, $\Delta_0 G_0(\xi, \eta) = 0$ for $\xi \neq \eta$. We see that the last two terms in (6.5) are canceled. Thus,

$$s_{\tilde{g}_\Gamma, \mathbb{Q}} = 2(Q+2) \left(\frac{Q}{4} - \frac{\delta+1}{2} \right) e^{-2f} \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-2} \sum_{j=1}^{4n} |(Y_j \phi_\eta)|^2 d\nu - \sum_{j=1}^{4n} \left(\int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \right)^2 \right).$$

Let $A = (A_{jk})$ be the symmetric matrix with

$$(6.6) \quad A_{jk} = \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta \cdot \phi_\eta^{-1} Y_k \phi_\eta d\nu - \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_j \phi_\eta d\nu \int_{\tilde{\Lambda}(\Gamma)} \phi_\eta^{-1} Y_k \phi_\eta d\nu$$

Note that for real number a_j , $j = 1, \dots, 4n$, we have

$$\sum_{j,k=1}^{4n} A_{jk} a_j a_k = \int_{\tilde{\Lambda}(\Gamma)} h^2 d\nu - \left(\int_{\tilde{\Lambda}(\Gamma)} h d\nu \right)^2,$$

with $h = \sum_{j=1}^{4n} a_j \phi_\eta^{-1} Y_j \phi_\eta$. So it is easy to see that A is non-negative by the Cauchy-Schwarz inequality and $\int_{\tilde{\Lambda}(\Gamma)} d\nu = 1$, i.e. $A(\xi) \geq 0$. Then,

$$(6.7) \quad s_{\tilde{g}_\Gamma, \mathbb{Q}} = 8(n+2) \left(n+1 - \frac{\delta}{2} \right) e^{-2f} \text{Tr} A(\xi).$$

Hence, the result follows from the fact that $\text{Tr} A(\xi)$ is nowhere vanishing on $\Omega(\Gamma)$ by the following lemma. \square

Lemma 6.1. *If $\text{Tr} A(\xi)$ vanishes at some point $\xi \in \tilde{\Omega}(\Gamma)$ and $\delta(\Gamma) \neq 2n+2$, then the limit set $\tilde{\Lambda}(\Gamma)$ is exactly one point, and A vanishes identically.*

The proof is exactly the same as that in the CR case (cf. Lemma 4.1 in [33]).

APPENDIX A. THE GREEN FUNCTION OF THE QC YAMABE OPERATOR ON THE QUATERNIONIC HEISENBERG GROUP

See also §2.4 in [15] for a similar calculation for regular solutions of the qc Yamabe equation.

Proof of Proposition 3.4. Without loss of generality, we assume $\xi = 0$. Denote $\eta = (y, t)$. Recall that $\Delta_0 = -\frac{1}{2} \sum_{j=1}^{4n} Y_j Y_j$. We have

$$(A.1) \quad Y_{4l+j} \left(\frac{1}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q-2}{4}}} \right) = -\frac{Q-2}{4} \frac{Y_{4l+j} \|\eta\|^4}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q+2}{4}}},$$

with

$$(A.2) \quad Y_{4l+j} \|\eta\|^4 = 4|y|^2 y_{4l+j} + 4 \sum_{s=1}^3 \sum_{k=1}^4 b_{kj}^s y_{4l+k} t_s,$$

by using the expression of the vector field Y_{4l+j} in (2.17). Note that $\frac{1}{\sqrt{2}}Y_j$ is an orthonormal basis. Then we get

$$(A.3) \quad \Delta_0 \left(\frac{1}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q-2}{4}}} \right) = -\frac{Q-2}{4} \left[\frac{\Delta_0 \|\eta\|^4}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q+2}{4}}} + \frac{(Q+2)}{4} \frac{|\nabla_0 \|\eta\|^4|^2}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q+6}{4}}} \right],$$

where

$$(A.4) \quad \begin{aligned} 2|\nabla_0 \|\eta\|^4|^2 &= \sum_{l=0}^{n-1} \sum_{j=1}^4 (Y_{4l+j} \|\eta\|^4)^2 \\ &= 16 \sum_{l=0}^{n-1} \sum_{j=1}^4 \left(|y|^4 y_{4l+j}^2 + \sum_{s,s'=1}^3 \sum_{k,k'=1}^4 b_{k'j}^{s'} b_{kj}^s y_{4l+k'} y_{4l+k} t_s t_{s'} \right) = 16 \|\eta\|^4 |y|^2, \end{aligned}$$

by using (A.2) and $\sum_{j=1}^4 b_{kj}^s b_{jk'}^{s'} = (b^s b^{s'})_{kk'}$, antisymmetry for $b^s b^{s'}$ of $s \neq s'$ and $(b^s)^2 = -\text{id}$. Similarly, by (A.2), we get

$$(A.5) \quad \Delta_0 \|\eta\|^4 = -\frac{1}{2} \sum_{l=0}^{n-1} \sum_{j=1}^4 \left(8y_{4l+j}^2 + 4|y|^2 + 8 \sum_{s=1}^3 \sum_{k,k'=1}^4 b_{k'j}^s b_{kj}^s y_{4l+k'} y_{4l+k} \right) = -2(Q+2)|y|^2.$$

Then apply (A.4) and (A.5) to (A.3) to get

$$(A.6) \quad \Delta_0 \left(\frac{1}{(\|\eta\|^4 + \epsilon^2)^{\frac{Q-2}{4}}} \right) = \frac{(Q-2)(Q+2)|y|^2 \epsilon^2}{2(\|\eta\|^4 + \epsilon^2)^{\frac{Q+6}{4}}}.$$

Now letting $\epsilon \rightarrow 0$, we see that for any $u \in C_0^\infty(\mathcal{H})$,

$$\begin{aligned} \int_{R^{4n+3}} L_0 u \cdot \frac{C_Q}{(|y|^4 + |t|^2)^{\frac{Q-2}{4}}} &= \lim_{\epsilon \rightarrow 0} \int_{R^{4n+3}} L_0 u \cdot \frac{C_Q}{(|y|^4 + |t|^2 + \epsilon^2)^{\frac{Q-2}{4}}} \\ &= \lim_{\epsilon \rightarrow 0} \int_{R^{4n+3}} u \cdot C_Q b_n \Delta_0 \left(\frac{1}{(|y|^4 + |t|^2 + \epsilon^2)^{\frac{Q-2}{4}}} \right) = u(0) \end{aligned}$$

by (A.6) and the formula (3.12) for C_Q^{-1} . The proposition is proved. \square

For $g|_\xi = \frac{g_0|_\xi}{\|\xi\|^2}$, we can write $g = \phi^{\frac{4}{Q-2}} g_0$ with $\phi = \|\xi\|^{-\frac{Q-2}{2}}$. It follows from the transformation law of the scalar curvature (3.6) that

$$(A.7) \quad \begin{aligned} s_{g,\mathbb{Q}} &= \phi^{-\frac{Q+2}{2}} b_n \Delta_0 \phi = \|\xi\|^{\frac{Q+2}{2}} b_n \Delta_0 \|\xi\|^{-\frac{Q-2}{2}} \\ &= -\frac{Q-2}{8} b_n \|\xi\|^{\frac{Q+2}{2}} \left(\frac{Q+6}{8} \|\xi\|^{-\frac{Q+14}{2}} |\nabla_0 \|\xi\|^4|^2 + \|\xi\|^{-\frac{Q+6}{2}} \Delta_0 \|\xi\|^4 \right) \\ &= \frac{(Q-2)(Q+2)}{2} \frac{|y|^2}{\|\xi\|^2} \end{aligned}$$

by (A.4) and (A.5), where $\eta = (y, t)$. Similarly, we have

$$(A.8) \quad \left| \nabla_g \left(\ln \frac{1}{\|\xi\|} \right) \right|^2 = \|\xi\|^2 \left| \nabla_0 \left(\ln \frac{1}{\|\xi\|} \right) \right|^2 = \frac{1}{16\|\xi\|^4} |\nabla_0 \|\xi\|^4|^2 = \frac{1}{2} |y|^2.$$

REFERENCES

- [1] Alt, J., Weyl connections and the local sphere theorem for quaternionic contact structures, *Ann. Global Anal. Geom.* **39** (2011), 165-186.
- [2] Apanasov, B., and Kim, I., Cartan angular invariant and deformations of rank 1 symmetric spaces, *Sb. Math.* **198**(2) (2007), 147-169.
- [3] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, *Astérisque* **265** (2000).
- [4] Cheng, J., Chiu, H. and Yang, P., Uniformization of spherical CR manifolds, *Adv. Math.* **255** (2014), 182-216.
- [5] Conti, D., Fernández, M., Santisteban, J. and José, A., On seven-dimensional quaternionic contact solvable Lie groups, *Forum Math.* **26** (2014), 547-576.
- [6] Corlette, K., Hausdorff dimension of limit sets I, *Invent. Math.* **102** (1990), 521-541.
- [7] Duchemin, D., Quaternionic contact structures in dimension 7, *Ann. Inst. Fourier. Grenoble* **56** (2006), 851-885.
- [8] de Andrés, L., Fernández, M., Ivanov, S., Santisteban, J., Ugarte, L. and Vassilev, D., Quaternionic Kähler and *Spin*(7) metrics arising from quaternionic contact Einstein structures, *Ann. Mat. Pura Appl.* **193** (2014), 261-290.
- [9] Eberlein, P.O., Visibility Manifold, *Pac. J. Math.* **46** (1973), 45-109.
- [10] Epstein, C., Melrose, R. and Mendoza, G., Resolvent of the Laplacian on strictly pseudoconvex domains, *Acta. Math.* **167** (1991), 1-106.
- [11] Habermann, L. and Jost, J., Green functions and conformal geometry, *J. Differ. Geom.* **53** (1999), 405-442.
- [12] Habermann, L., *Riemannian metrics of constant mass and moduli spaces of conformal structures*, *Lect. Notes Math.* **1743**, Berlin, Springer (2000).
- [13] Ivanov, S. and Vassilev, D., Conformal quaternionic contact curvature and the local sphere theorem, *J. Math. Pures Appl.* **93** (2010), 277-307.
- [14] Ivanov, S. and Vassilev, D., Quaternionic contact manifolds with a closed fundamental 4-form, *Bull. London Math. Soc.* **42** (2010), 1021-1030.
- [15] Ivanov, S. and Vassilev, D., *Extremals for the Sobolev Inequality and the quaternionic contact Yamabe Problem*, New Jersey, London, World Scientific, (2011).
- [16] Ivanov, S. Minchev, I. and Vassilev, D., Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem, *Mem. Amer. Math. Soc.* **231** (2014).
- [17] Ivanov, S. and Vassilev, D., The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds, *Nonlinear Analysis.* **126** (2015), 262-323.
- [18] Ivanov, S. Minchev, I. and Vassilev, D., Solution of the qc Yamabe equation on a 3-Sasakian manifold and the quaternionic Heisenberg group, arXiv:1504.03142v1.
- [19] Ize, H., Limits sets of Kleinian groups and conformally flat Riemannian manifolds, *Invent. Math.* **122** (1995), 603-625.
- [20] Ize, H., The Teichmüller distance on the space of flat conformal structures, *Conform Geom Dyn.* **2** (1998), 1-24.
- [21] Jerison, D. and Lee, J. M., The Yamabe problem on CR manifolds, *J. Diff. Geom.* **25**, (1987), 167-197.
- [22] Kaplan, A., Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, *Trans. Amer. Math. soc.* **258** (1980), 147-153.
- [23] Kobayashi, O., Scalar curvature of a metric with unit volume, *Math. Ann.* **279** (1987), 253-265.
- [24] Leutwiler, H., *A Riemannian metric invariant under Möbius transformations in \mathbb{R}^n* , *Lect. Notes in Math.* **1351**, Berlin, Springer (1988), 223-235.
- [25] Li, Z., Uniformization of spherical CR manifolds and the CR Yamabe problem, *Proc. Symp. Pure Math.* **54**, Part 1 (1993), 299-305.
- [26] Nayatani, S., Patterson-Sullivan measure and conformally flat metrics, *Math. Z.* **225** (1997), 115-131.
- [27] Nayatani, S., Discrete groups of complex hyperbolic isometries and pseudo-Hermitian structures, *Analysis and geometry in several complex variables*, Boston, Birkhäuser, (1997), 209-237.
- [28] Orsted, B., Conformally invariant differential equations and projective geometry, *J. Funct. Anal.* **44** (1981), 1-23.

- [29] Pansu, P., Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, *Ann. of Math.* **129** (1989), 1-60.
- [30] Patterson, S.J., The limit set of a Fuchsian group, *Acta. Math.* **136** (1976), 241-273.
- [31] Salamon, S., *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Mathematics **201**, Longman, Harlow, (1989).
- [32] Schoen, R. and Yau, S., Conformally flat manifolds, Kleinian groups and scalar curvature, *Invent. Math.* **92** (1988), 47-71.
- [33] Wang, W., Canonical contact forms on spherical CR manifolds, *J. Eur. Math. soc.* **5** (2003), 245-273.
- [34] Wang, W., Representations of $SU(p, q)$ and CR geometry I, *J. Math. Kyoto Univ.* **45** (2005), 759-780.
- [35] Wang, W., The Teichmüller distance on the space of spherical CR structures, *Sci. China Ser. A.* **49** (2006), 1523-1538.
- [36] Wang, W., The Yamabe problem on quaternionic contact manifolds, *Ann. Mat. Pura Appl.* **186** (2007), 359-380.
- [37] Wang, W., On the tangential Cauchy-Fueter operators on nondegenerate quadratic hypersurfaces in \mathbb{H}^2 , *Math. Nachr.* **286** (2013), 1353-1376.
- [38] Yue, C., Mostow rigidity of rank 1 discrete groups with ergodic Bowen-Margulis measure, *Invent. Math.* **125** (1996), 75-102.